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Optimal Stochastic Control of life insurance and investment in a financial market

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September 2013

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Thesis submitted to the Faculty of Sciences
of the University of Porto, to obtain the degree of
Master of Science in Mathematics,
under the supervision of Professor Alberto Adrego Pinto.

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Acknowledgements

I would like to thank Alberto Adrego Pinto for proposing me this subject, as well as others, and for all his advisement and guidance in this work, and for his friendship.

I also would like to thank Diogo Pinheiro, for all suggestions and commentaries.

I would like to thank all my family, in particular my parents.

Finally, I would like to thank all my friends, from the ones that I know only for the last few years, to the others with whom I have the most enduring friendship since the 1990's. Thank you all! There's no life without friends.

Abstract

In this work we analyse a consumption, investment and life insurance purchase problem, in a very general model of a financial market with stochastic coefficients that are not necessarily Markov processes, and that we assume to be complete. We use duality tools from convex analysis to obtain optimal consumption, investment and life insurance purchase under very general utility functions. We analyse the case of deterministic coefficients, deducing a mutual fund result, and the Hamilton-Jacobi-Bellman equation on that case, and we obtain explicit solutions for utility functions with constant relative risk aversion (CRRA).

Keywords: Consumption-investment problems; life insurance; convex duality

Resumo

Neste trabalho analisamos um problema de consumo, investimento e compra de seguro de vida num modelo muito geral de um mercado financeiro com coeficientes estocásticos não necessariamente Markov, e que assumiremos completo. Usamos ferramentas de dualidade da análise convexa para obter estratégias óptimas de consumo, investimento e compra de seguro de vida para funções de utilidade muito gerais. Analisamos o caso em que os coeficientes são determinísticos, deduzimos um resultado do tipo 'mutual fund', e derivamos a equação de Hamilton-Jacobi-Bellman para esse caso, e obtemos soluções explícitas para funções de utilidade com índice relativo de aversão ao risco constante (CRRA).

Palavras-chave: Problemas de Consumo-Investimento; seguro de vida; dualidade convexa

Contents

Chapter 1

Introduction

In this work we consider the problem faced by a wage earner having to make decisions continuously about three strategies: consumption, investment and life insurance purchase, during a given interval of time $[0, \min\{\tau, T\}]$, where T is a fixed point in the future that we will see as the retirement date of the wage earner, and τ is a random variable representing the wage earner's death time of death. We assume that the wage earner receives income that is terminated when he dies or retires, whichever happens first. One of our main assumptions is that the wage earner's lifetime τ is a random variable, and, therefore, he needs to buy life insurance to protect his family for the eventuality of premature death. This life insurance depends on an insurance premium payment rate $p(t)$ such that if the insured pays $p(t)\delta(t)$ and dies during the ensuing short-time interval of length $\delta(t)$ then the insurance company will pay $p(t)/\eta(t)$ dollars to the insured's estate, where $\eta(t)$ is an amount set in advance by the insurance company. Hence, this is like term insurance with an infinitesimal term. We also assume that the wage earner wants to maximize the satisfaction obtained from a consumption process with rate $c(t)$. In addition to consumption and purchase of a life insurance policy, we assume that the wage earner invests the full amount of his savings in a complete financial market consisting of one money market and a arbitrary finite number of risky securities with diffusive terms driven by a multi-dimensional Brownian motion. The financial market is quite general and complex, with all the coefficients being stochastic processes, that are not assumed to be Markovian. All parameter with the exception of the hazard rate can be stochastic processes.

The wage earner has to find strategies that maximize the utility of (i) his family consumption for all $t \leq \min\{\tau, T\}$;(ii) his wealth at retirement date T if he lives that long and (iii) the value of his estate in the event of premature death. Various quantitative models models have been proposed to model and analyse problems having at least one of these three objectives. Yarri's paper [13] considered the problem of optimal financial planning decisions for an individual with an uncertain lifetime. This paper is generally considered to be the starting point of research on demand for life insurance by introducing an uncertain lifetime. Later, Merton, in his celebrated papers [6, 7] emphasized optimal consumption and investment decisions without considering life insurance in both a finite and infinite time horizon. Richard [10] and analyses a life-cycle life insurance and consumption investment problem in a continuous time model by combining Yarri's uncertain lifetime setting and Merton's dynamic programming approach. Later, Pliska and Ye [9], generalized the previously considered models, combining their more realistic features and considering a different boundary condition, that lead to somewhat different economic interpretations than the ones provided by Richard. Pliska and Ye's paper considered only one risky security, and considered that the market was complete. The main difference between Richard's paper and Pliska and Ye's paper is that while Richard considered that the lifetime of the wage earner is limited by some fixed number, Pliska and Ye considered that the lifetime of the individual is a random variable taking positive values and that it is independent of the stochastic process associated to the underlying financial market, and the fixed horizon T is now seen as the moment when the wage earner retires. Duarte, Pinheiro, Pinto and Pliska [1] extended Pliska and Ye's paper to a more general setting where there is an arbitrary (finite) number of risky securities, and without necessarily assuming completeness of the market.

All the previously cited literature adopts a dynamical programming approach that leads to an associated Hamilton-Jacobi-Bellman equation, which enables the characterization (and in some cases, computation) of the optimal feedback controls, that is, the optimal life insurance, portfolio and consumption strategies. This dynamic programming approach was valid due to the construction of the underlying financial market, where the coefficients were deterministic functions, which led to Markovian processes, and thus permitted to state a dynamic programming principle. In [4], Karatzas and Shreve propose a very general financial market model where the coefficients are stochastic processes that are not assumed to be Markovian, and in section 3 of the same reference, they study the problem of utility maximization from both consumption and terminal wealth in a complete market without considering insurance and without considering any income apart from initial wealth. Due to the generality of the market, and the consideration of very general utility functions, the analysis relied on duality methods of convex analysis, rather than involving dynamic programming. In this work, we extend this analysis to the case including the possibility of buying life insurance and we adopt Pliska and Ye's formulation for the life insurance market.

Both the duality approach and the dynamic programming approach have strengths and weaknesses. The first one is valid in very general setting, which an advantage, but, as a result, provides more general solutions to the problem, that are harder to analyse, and also, from the mathematical point of view, it is less standard than the very frequently used dynamic programming approach, as we mentioned. On the other hand, the dynamic programming approach is not good when we are dealing with stochastic parameters. However, being based around the Hamilton-Jacobi-Bellman equation, and as one has optimality of solution intimately related to the Hamilton-Jacobi-Bellman equation, one can use several powerful numerical methods that are available for HJB equations.

This thesis is organized in the following way. In chapter 2, we describe and solve the main problem we propose to address. We introduce the underlying financial and insurance markets, the wealth process, the utility functions, and the optimization problems. We expose the relevant mathematical tools from stochastic calculus that we will use in the analysis that follows, and we provide references for the proofs of those results. We discuss admissibility of the controls, and restate the original problem as one with a fixed planning horizon and derive the general form of the optimal strategies. In chapter 3, we restrict to the case of deterministic coefficients, and on that case we are able to deduce more about the very general optimal solutions obtained in chapter 2, deducing feedback form solutions, and deriving a mutual fund result. We then obtained for the case of deterministic coefficients, the Hamilton-Jacobi-Bellman equation of dynamic programming. We derive explicit optimal solutions for the case of CRRA (constant relative risk aversion) utilities (power and logarithmic). We conclude and address possible future research topics in chapter 4.

Chapter 2

Optimal life insurance purchase, consumption and investment

2.1 Problem Formulation

In this section, we define the setting where the agent makes decisions regarding his consumption, investment and life insurance purchase. We define the financial and insurance markets that are available to him, as well as his wealth process, his utility functions and subsequent optimization problems

2.1.1 The Financial Market Model

The financial market model that we will consider is described with detail in chapter 1 of [4].

We start by considering a complete probability space $(\Omega, \mathcal{F}, P)$, on which a D-dimensional Brownian motion $W(t) = (W_1(t), \ldots, W_D(t))'$ is defined, where the prime denotes transposition, for $t \in [0, T]$, for some positive constant T. We assume that $W(0) = 0$ almost surely (a.s.). We define

$$\mathcal{F}^W(t) = \sigma\{W(s), 0 \leq s \leq t\}$$

to be the canonical filtration generated by the Brownian Motion, for $t \in [0, T]$. We denote by $\mathcal{N}$ the P-null subsets of $\mathcal{F}^W(T)$, and we consider the filtration $\{\mathcal{F}(t)\}$ given by

$$\mathcal{F}(t) = \sigma\left(\mathcal{F}^W(t) \cup \mathcal{N}\right) \quad t \in [0, T]$$

and we call it the *P-augmented filtration*. We will use the *P-augmented filtration* for technical reasons that we now explain. The *P-augmented filtration* is both *left-continuous* and *right-continuous*, in the sense that it verifies both

$$\mathcal{F}(t) = \sigma\left(\bigcup_{0 \leq s < t} \mathcal{F}(s)\right), \quad t \in (0, T]$$

and

$$\mathcal{F}(t) = \bigcap_{t < s \leq T} \mathcal{F}(s), \quad t \in [0, T)$$

respectively, while the filtration $\{\mathcal{F}^W(t)\}_{0 \leq t \leq T}$ doesn't satisfy the second property. More details about these properties can be seen in [3].

The filtration naturally represents the flow of information, namely, $\mathcal{F}(t)$ can be interpreted as the information available to investors at time t. The two properties mentioned before can be interpreted as meaning that "there are no surprises in the flow of information". We shall always define progressive measurability of processes with respect to this filtration.

The financial market that we consider is composed of a *money market* and an arbitrary finite number of *stocks*.

The price of one share of the *money market* at time t is denoted by $S_0(t)$ and we will assume, for simplicity, that $S_0(0) = 1$. We assume that the process $S_0(\cdot)$ is continuous, strictly positive, $\{\mathcal{F}(t)\}$-adapted and has finite total variation on $[0, T]$. Being of finite variation, it can be decomposed into absolutely continuous and singularly continuous parts, that we will denote respectively by $S_0^{ac}(\cdot)$ and $S_0^{sc}(\cdot)$. Se we can define

$$r(t) = \frac{\frac{d}{dt} S_0^{ac}(t)}{S_0(t)}, \quad A(t) = \int_0^t \frac{S_0^{sc}(u)}{S_0(u)} \tag{2.1}$$

and we can write

$$dS_0(t) = S_0(t)(r(t)dt + dA(t)),$$

and equivalently

$$S_0(t) = \exp\left\{\int_0^t r(u)du + A(t)\right\}$$

In the special case that $S_0(\cdot)$ is itself absolutely continuous, so $A(\cdot) = 0$, and the price of the money market evolves like the value of a savings account whose instantaneous *(risk-free) interest rate* at time t is $r(t)$. This situation is very common, and provides the usual interpretation of the money market that one should have in mind, although the singular continuous part doesn't necessarily need to be zero. We have that $r(\cdot)$ is a stochastic process, but $r(t)$ is $\mathcal{F}(t)$-measurable, so the current time risk-free rate is known to all investors.

The price of a certain stocks is denoted by $(S_n(t))_{0\leq t\leq T}$, for $n = 1, \ldots, N$, and it evolves according to the following stochastic differential equation:

$$dS_n(t) = S_n(t)\left(\mu_n(t)dt + dA(t) + \textstyle\sum_{d=1}^{N} \sigma_{nd} dW^{(d)}(t)\right)$$

It can be shown that every continuous, strictly positive and $\{\mathcal{F}(t)\}$-adapted semimartingale satisfies a stochastic differential equation of this form, where $A(\cdot)$ is $\{\mathcal{F}(t)\}$-adapted and singularly continuous. However, in the previous equation $A(\cdot)$ is not an arbitrary process with these properties, but it is actually, the one defined before in (2.1). This happens because if $A(\cdot)$ is not given by (2.1), then there would be an *arbitrage opportunity.* The proofs of these results can be found in appendix B of [4].

So we can now formalize this discussion with the following definition

Definition 1. *A financial market consists of*

1. *a complete probability space* $(\Omega, \mathcal{F}, P)$*;*
2. *a positive constant* T*, called the* ***terminal time****;*
3. *a D-dimensional Brownian Motion* $\{W(t), \mathcal{F}(t); 0 \leq t \leq T\}$ *defined on* $(\Omega, \mathcal{F}, P)$*, where* $\{\mathcal{F}(t)\}$ *is the P-augmentation of the canonical filtration generated by the Brownian motion;*
4. *a progressively measurable process* $r(\cdot)$*, called the* ***risk-free rate process****, satisfying*
$$\int_0^T |r(t)|dt < \infty, \quad a.s.;$$
5. *a progressively measurable, singularly continuous, finite variation process* $A(\cdot)$*;*

6. a progressively measurable process $\mu(\cdot) = (\mu_1(\cdot), \ldots, \mu_N(\cdot))'$, called the ***mean rate of return process****, satisfying*

$$\int_0^T ||\mu(t)||dt < \infty, \quad a.s.;$$

7. a progressively measurable process $\delta(\cdot) = (\delta_1(\cdot), \ldots, \delta(\cdot))'$ called the ***dividend rate process****, that verifies*

$$\int_0^T ||\delta(t)||dt < \infty, \quad a.s.;$$

8. a $(N \times D)$-matrix-valued process $\sigma(\cdot)$, called the ***volatility process****, that satisfies*

$$\sum_{n=1}^{N} \sum_{d=1}^{D} \int_0^T \sigma_{nd}^2(t)dt < \infty, \quad a.s.;$$

9. a strictly positive, constant vector of ***initial stock prices*** $S(0) = (S_1(0), \ldots, S_N(0))'$

We denote this financial market by $\mathcal{M} = (r(\cdot), \mu(\cdot), \delta(\cdot), \sigma(\cdot), S(0), A(\cdot))$.

Furthermore, we suppose that there exists a process $\theta(\cdot)$, called the *market price of risk* such that for Lebesgue-almost-every $t \in [0, T]$, the *risk premium*

$$\mu(t) + \delta(t) - r(t)\underline{1}$$

is related to $\theta(t)$ by the equation

$$\mu(t) + \delta(t) - r(t)\underline{1} = \sigma(t)\theta(t), \quad a.s.$$

where $\underline{1}$ denotes the N-dimensional column vector with all components equal to one, and is such that the following two conditions holds:

$$||\theta||_2^2 = \int_0^T ||\theta(t)||^2 dt < \infty, \quad a.s.$$

$$E\left[\exp\left(-\int_0^T \theta'(s)dW(s) - \frac{1}{2}\int_0^T ||\theta(t)||^2 ds\right)\right] = 1$$

The existence of θ with these condition ensures that there are no *arbitrage opportunities* in the financial market. We will further assume that the market is *complete.* A necessary and sufficient condition for the market to be complete is that the number of stocks N equals the dimension D of the underlying Brownian motion and that the volatility matrix is non-singular for Lebesgue-a.e. $t \in [0,T]$, almost surely. We shall assume these conditions from now on. Under completeness of the market, the market price of risk is uniquely determined by $\theta(t) = (\sigma(t))^{-1}(\mu(t) + \delta(t) - r(t)\underline{1}), 0 \le t \le T$. For proofs of these results we again refer to [4].

We also suppose that we have a *standard financial market*, by further assuming that the positive local martingale

$$Z_0(t) = \exp\left(-\int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds\right), 0 \le t \le T$$

is indeed a martingale. We define the *standard martingale measure* P_0 (or *risk neutral probability measure*) on $\mathcal{F}(T)$ by

$$P_0(A) = E[Z_0(T)1_A], A \in \mathcal{F}(T)$$

where 1_A denotes the indicator function of the set A. Note that the probability measures P and P_0 are equivalent on $\mathcal{F}(T)$, i.e., they have the same null-sets on $\mathcal{F}(T)$.

A sufficient condition for $Z_0(\cdot)$ to be a martingale is the *Novikov condition*:

$$E\left[\exp\left\{\frac{1}{2}\int_0^T ||\theta(t)||^2 dt\right\}\right] < \infty$$

According to Girsanov theorem, (see [3], section 3.5), the process $W_0(t)$, for $0 \le t \le T$ defined by

$$W_0(t) = W(t) + \int_0^t \theta(s)ds$$

is a Brownian motion under P_0 relative to the filtration $\{\mathcal{F}(t)\}$.

We will denote expectation with respect to the martingale measure P_0 by E_0. We shall make frequent use of the following theorem, known as Bayes's rule, that provides a way to change between expectation with respect to P_0 and expectation with respect to the original probability measure P. See lemma 3.5.3 in [3] for a proof.

Lemma (Bayes's rule)**.** *For $0 \le s \le t \le T$, let Y be a $\mathcal{F}(t)$-measurable random variable satisfying $E_0(|Y|) < \infty$. Then*

$$E_0\left(Y|\mathcal{F}(s)\right) = \frac{1}{Z_0(s)} E\left(Y Z_0(t)|\mathcal{F}(s)\right)$$

We define the *state price density*

$$H_0(t) = \frac{Z_0(t)}{S_0(t)}$$

We will assume that it satisfies the following condition

Assumption 1.

$$E\left[\int_0^T H_0(t)dt + H_0(T)\right] < \infty$$

2.1.2 The life insurance market model

We suppose that the agent is alive at time $t = 0$ and that his lifetime is a non-negative random variable τ defined on the probability space $(\Omega, \mathcal{F}, P)$. Furthermore, we assume that the random random variable τ is independent of the filtration $\{\mathcal{F}(t)\}$ and has a distribution function $F : [0, \infty) \to [0, 1]$ with density function $f : [0, \infty) \to \mathbb{R}^+$ so that

$$F(t) = P(\tau < t) = \int_0^t f(s)ds.$$

We define the *survivor function* $\bar{F} : [0, \infty) \to [0, 1]$ as the probability for the agent to survive at least until time t, i.e.

$$\bar{F}(t) = P(\tau \geq t) = 1 - F(t).$$

We shall make use of the *hazard function*, the conditional, instantaneous death rate for the agent surviving to time t, that is

$$\lambda(t) = \lim_{\delta t \to 0} \frac{P(t \leq \tau \leq t + \delta t | \tau \geq t)}{\delta t} = \frac{f(t)}{\bar{F}(t)}$$

Throughout this article, we will suppose that the hazard function $\lambda : [0, \infty) \to \mathbb{R}^+$ is a continuous and deterministic function such that

$$\int_0^\infty \lambda(t)dt = \infty$$

With these assumptions, we have

$$\bar{F}(t) = \exp\left(-\int_0^t \lambda(u)du\right), \quad f(t) = \lambda(t)\bar{F}(t) \tag{2.2}$$

These two concepts introduced above are standard in the context of reliability theory and actuarial science. In our case, such concepts enable us to consider an optimal control problem with a stochastic planning horizon and restate it as one with a fixed horizon.

Due to uncertainty concerning his life time, the agent buys life insurance to protect his family for the eventuality of premature death. The life insurance is available continuously and the agent buys it by paying an *insurance premium payment rate* *p(t)* to the insurance company. The insurance contract is like term insurance with an infinitesimally small term. If the agent dies at time $\tau < T$, while buying insurance at the rate $p(t)$, the insurance company pays an amount $p(\tau)/\eta(\tau)$ to his estate, where $\eta(\cdot)$ is a nonnegative, progressively measurable, almost surely uniformly bounded process, called *insurance premium-payout ratio* and is regarded as fixed by the insurance company. The contract ends when the agent dies or achieves retirement age, whichever happens first. Therefore, the agent's total legacy to his estate in the event of premature death at time $\tau < T$ is given by

$$Z(\tau) = X(\tau) + \frac{p(\tau)}{\eta(\tau)} \tag{2.3}$$

where $X(t)$ denote the agent's savings at time t.

2.1.3 The wealth process

We now define the actions that the agent can take and the respective wealth process that arises from those decisions.

We assume that the agent is endowed with the *initial wealth* x, and that he receives *income* $i(t)$ during the period $[0, \min\{(T, \tau)]$, i.e., the income will be terminated either by his death or his retirement, whichever happens first. We assume that $i(\cdot)$ is a nonnegative, progressively measurable process, that satisfies

$$\int_0^T i(t)dt < \infty, \quad a.s.$$

The *consumption process* $(c(t))_{0 \le t \le T}$ is a progressively measurable nonnegative process satisfying

$$\int_0^T c(t)dt < \infty, \quad a.s.$$

We also assume that the *insurance premium payment rate* $(p(t))_{0 \le t \le T}$ is a predictable process (see [3]), that satisfies

$$\int_0^T |p(t)|dt < \infty, \quad a.s.$$

In an intuitive manner, a predictable process can be described as such that its values are 'known' just in advance of time.

Under these hypothesis, the agent has a *cumulative income process*

$$\Gamma(t) = x - \int_0^t c(u) + p(u) - i(u)du, \quad 0 \le t \le \min\{\tau, T\}$$

We define the *portfolio process* $\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_N(\cdot))$ such that $\pi_n(t)$ is the *dollar amount* invested in security n at time t. We assume that the portfolio is $\Gamma(\cdot)$-*financed*, which means that the agent invests the full amount of his savings in the financial market, that is

$$X(t) = \pi_0(t) + \pi(t)\underline{1} \tag{2.4}$$

where $\pi_0(t)$ is the dollar amount invested in the money market at time t. Observe that we need only specify $\pi(t)$, as we can obtain $\pi_0(t)$ by imposing equation (2.4), and because of this feature we will refer to $\pi(\cdot)$ alone as a portfolio process. We assume that this process is progressively measurable and that it satisfies the following integrability conditions

$$\int_0^T |\pi'(t)(\mu(t) + \delta(t) - r(t)\underline{1})|dt < \infty, \quad a.s.$$
$$\int_0^T ||\sigma'(t)\pi(t)||^2 dt < \infty, \quad a.s.$$

We define the *discounted wealth process* $X(t)$, $0 \le t \le \min\{\tau, T\}$

$$\frac{X(t)}{S_0(t)} = x - \int_0^t \frac{c(u) + p(u) - i(u)du}{S_0(u)} + \int_0^t \frac{1}{S_0(u)}\pi'(u)\sigma(u)dW_0(u) \tag{2.5}$$

Using relation (2.3) we may write

$$\frac{X(t)}{S_0(t)} = x - \int_0^t \frac{c(t) + \eta(t)Z(t) - \eta(t)X(t) - i(t)}{S_0(t)}dt + \int_0^t \frac{1}{S_0(t)}\pi'(t)\sigma(t)dW_0(t)$$

If we let $D(t) = \exp\left(\int_0^t \eta(u)du\right)$, applying Itô's lemma to the product $\frac{1}{D(t)}\frac{X(t)}{S_0(t)}$, we obtain

$$\frac{X(t)}{D(t)S_0(t)} + \int_0^t \frac{c(u) + \eta(u)Z(u)}{D(u)S_0(u)}du = x + \int_0^t \frac{i(u)}{D(u)S_0(u)}du +$$
$$+ \int_0^t \frac{1}{D(u)S_0(u)}\pi'(u)\sigma(u)dW_0(u) \tag{2.6}$$

2.1.4 Utility functions and preference structures

We will impose very general conditions on the utility functions of the agent and that are adapted from [4] to suit our purposes.

A *utility function* is a function $U : \mathbb{R} \to [-\infty, \infty)$ that is concave, non-decreasing, upper semi-continuous, such that

1. $\bar{x} = \inf\{x \in \mathbb{R} : U(x) > -\infty\} \geq 0$ and $\bar{x} < \infty$
2. U' is continuous, positive, strictly decreasing on $(\bar{x}, \infty)$ and $U'(\infty) = \lim_{x\to\infty} U'(x) = 0$, $U'(\bar{x}+) = \lim_{x\downarrow\bar{x}} U'(x)$

Under these conditions, the function U' has a a continuous, strictly decreasing inverse $I : (0, U'(\bar{x}+)) \to (\bar{x}, \infty)$, that can be extended to the whole half line $(0, \infty]$, if we set $I(y) = \bar{x}$, for $y \in [U'(\bar{x}+), \infty]$.

The *convex dual* of U is defined by

$$\widetilde{U}(y) = \sup_{x\in\mathbb{R}}\{U(x) - xy\}, y \in \mathbb{R}$$

Taking the derivative with respect to x of the expression in the sup, and setting it equal to zero, we find that $U'(x) = y$, so $I(y) = x$, and we have

$$\widetilde{U}(y) = U(I(y)) - yI(y), y > 0 \tag{2.7}$$

A *time-separable Von-Neumann-Morgenstern preference structure* is a triple (U_1, U_2, U_3), where $U_1, U_2 : [0, T] \times \mathbb{R} \to [-\infty, \infty)$ and $U_3 : \mathbb{R} \to [-\infty, \infty)$, such that

1. for fixed $t \in [0, T]$, $U_1(t, \cdot)$ and $U_2(t, \cdot)$ are utilities and
   (a) $\bar{c}(t) = \inf\{c \in \mathbb{R} : U(t, c) > -\infty\}$
   (b) $\bar{Z}(t) = \inf\{z \in \mathbb{R} : U(t, z) > -\infty\}$

   are continuous, for $0 \leq t \leq T$, with values in $[0, \infty)$, and called, respectively, *subsistence consumption* and *subsistence legacy in case of premature death*
2. (a) U_1 and U_1', where the derivative is taken with respect to the second component, are continuous on the set $D_1 = \{(t, c) \in [0, T] \times (0, \infty); c > \bar{c}(t)\}$
   (b) U_2 and U_2', where the derivative is taken with respect to the second component, are continuous on the set $D_2 = \{(t, z) \in [0, T] \times (0, \infty); z > \bar{Z}(t)\}$

3. U_3 is a utility and $\bar{x} = \inf\{x \in \mathbb{R} : U_3(x) > -\infty\}$ is called *subsistence terminal wealth*

For fixed $t \in [0, T]$, as $U_i(t, \cdot)$ is a utility function for $i = 1, 2$, we define its convex dual, and denote it by $\widetilde{U}_i(t, \cdot)$, and denote its inverse (which is continuous and strictly decreasing), by $I_i(t, \cdot)$ and relation (2.7) becomes $\widetilde{U}_i(t, y) = U_i(I_i(t, y)) - yI_i(t, y), y > 0$. We do the same for $U_3(\cdot)$. We extend $I_1(t, \cdot), I_2(t, \cdot)$ and $I_3(\cdot)$ to the whole half-line $(0, \infty]$ as we did before, so we have $I_1(t, \cdot) = \bar{c}(t)$ on $[U_1'(t, \bar{c}(t)+), \infty]$, $I_2(t, \cdot) = \bar{Z}(t)$ on $[U_2'(t, \bar{Z}(t)+), \infty]$ and $I_3(\cdot) = \bar{x}$ on $[U_3'(\bar{x}+), \infty]$.

Remark 1. *By continuity of $\bar{c}(t)$ and $\bar{Z}(t)$, we can choose $\hat{c}$ and $\hat{z}$ such that $\hat{c}, \hat{z} < \infty$ and $\hat{c} > \bar{x} \vee \max_{t\in[0,T]} \bar{c}(t)$ and $\hat{z} > \bar{x} \vee \max_{t\in[0,T]} \bar{z}(t)$*[1].

By construction of $\hat{c}$ and $\hat{z}$, we have

$$\int_0^T |U_1(t, \hat{c} + \hat{z})| + |U_2(t, \hat{c} + \hat{z})|dt + |U_3(\hat{c} + \hat{z})| < \infty$$

2.1.5 Optimization problem

The agent has the problem of finding the strategies that maximize the expected utilities obtain from three different situations, namely:

1. his family consumption on $[0, \min\{\tau, T\}]$
2. his wealth at the retirement date T, if he lives up to that age
3. the value of the legacy he leaves to his family in the case of premature death

We define the set of *admissible strategies*, $\mathcal{A}(x)$ for a given initial endowment x. We define $\mathcal{A}(x) = \emptyset$ if $x < 0$, and, for $x \geq 0$, as the set of triples (c, π, p) such that $X(t) + b(t) \geq 0$ and $Z(t) \geq 0$, almost surely, where

$$b(t) = S_0(t)E_0 \left[\int_t^T \frac{i(s)}{\exp(\int_t^s \eta(u)du)S_0(s)} ds \middle| \mathcal{F}(t) \right]. \tag{2.8}$$

We will refer to the quantity that corresponds to the expectation as *human capital*, following the nomenclature introduced by S.F. Richard in [10] for the deterministic counterpart of this quantity when the coefficients involved are deterministic functions. It can be interpreted as the fair discounted value of the wage earner's future income

[1]We use the notation $a \vee b = \max\{a, b\}$.

from time t to time T, where the discounting is made by both the money market, as usual, and by the insurance premium-payout rate $\eta(\cdot)$, that represents the cost of insurance. The quantity $X(t) + b(t)$ can be regarded as the full wealth (present wealth plus future income) of the wage earner at time t. In particular, Richard proves that if at a certain time $t = t^*$, the agent relinquishes his right to receive $i(t), t > t^*$, and instead receives $b(t)$ then his optimal policies will remain unchanged. By imposing $Z(t) \geq 0$ we have that the wage earner doesn't want to leave debts to his family in the event of premature death. In [4], the condition imposed on the wealth is just $X(t) \geq 0$ instead of the one we consider here because on that case the agent doesn't receive money, so if the wealth reaches zero, he is in a situation of bankruptcy and must cease all his economic activities. In our case, the wealth can actually be negative, but bounded from below by $-b(t)$ at time t, as he expects to receive income such that, in the future, he will have non negative wealth, so in this case, negative wealth doesn't mean that he must cease all his activities in the market. However, $X(t) + b(t) = 0$ means a situation of bankruptcy.

The wage earner's problem can be stated as follows: for given x, find a strategy $(c, \pi, p) \in \mathcal{A}_1(x)$ that maximizes the expected utility

$$V(x) = \sup_{(c,\pi,p)\in\mathcal{A}_1(x)} E\left[\int_0^{\tau\wedge T} U_1(s, c(s))ds + U_2(s, Z(s))1_{\{\tau\leq T\}} + U_3(X(T))1_{\{\tau>T\}}\right] \tag{2.9}$$

where $\tau \wedge T = \min\{\tau, T\}$ and

$$\mathcal{A}_1(x) = \Bigg\{(c, \pi, p) \in \mathcal{A}(x) : E\left(\int_0^{\tau\wedge T} \min\{0, U_1(s, c(s))\}ds \; + \right. \\ \left. + \min\{0, U_2(s, Z(s))\}1_{\{\tau\leq T\}} + \min\{0, U_3(X(T))\}1_{\{\tau>T\}}\right) > -\infty\Bigg\} \tag{2.10}$$

Remark 2. *The preference structure forces the following constraints for Lebesgue-almost-every $t \in [0, T]$, almost surely*

- $c(t) \geq \bar{c}(t)$, $Z(t) \geq \bar{Z}(t)$
- $X(T) \geq \bar{x}$

otherwise (2.9) would be $-\infty$. If $\mathcal{A}_1(x) = \emptyset$ we define $V(x) = -\infty$.

2.2 Optimal Strategies

In this section we develop some characteristics of admissibility, and restate the original problem as one with a fixed planning horizon. We then characterize the optimal strategies.

2.2.1 Admissibility

In this section we derive a *budget constraint* for the admissible strategies, and show that if one starts with consumption and random variable that satisfy that budget constraint, then it is possible to replicate the behaviour of those random variable by an admissible portfolio and life insurance policy, that we call *hedging strategies*.

The integral on the right side of (2.6) is a local martingale with respect to the martingale measure P_0, and if we take $(c, \pi, p) \in \mathcal{A}(x)$ and add $b(t)/D(t)S_0(t)$ to both sides, then we obtain

$$\frac{X(t)+b(t)}{D(t)S_0(t)} + \int_0^t \frac{c(u)+\eta(u)Z(u)}{D(u)S_0(u)}du = x + E_0\left[\int_0^T \frac{i(s)}{D(s)S_0(s)}ds \Big| \mathcal{F}(t)\right] + \\ + \int_0^t \frac{1}{D(u)S_0(u)}\pi'(u)\sigma(u)dW_0(u) \quad (2.11)$$

The right hand side is a local martingale with respect to the martingale measure P_0, and the left hand side is bounded from below by zero by admissibility of (c, π, p). Fatou's lemma implies that it is a supermartingale with respect to P_0, so, we obtain, for $t = T$

$$E_0\left[\frac{X(T)}{D(T)S_0(T)} + \int_0^T \frac{c(u)+\eta(u)Z(u)}{D(u)S_0(u)}du\right] \leq x + b(0)$$

Using Bayes' rule, we can express this expectation with respect to the original probability measure P, and obtain the *budget constraint*

$$E\left[\bar{H}_0(T)X(T) + \int_0^T \bar{H}_0(u)(c(u)+\eta(u)Z(u))du\right] \leq x + b(0) \qquad (2.12)$$

where $\bar{H}_0(t) = \frac{H_0(t)}{D(t)}$

Note that by the fact that $\eta(\cdot)$ is almost surely uniformly bounded, and assumption 1, by definition of $\bar{H}_0(t)$ we have

$$E\left[\int_0^T \bar{H}_0(t)dt + \bar{H}_0(T)\right] < \infty$$

Conversely, if we start with a consumption process and two non negative random variable satisfying the budget constrain, then there is a *hedging portfolio* and a *hedging life insurance policy* that lead to terminal wealth and legacy in event of death that are equal to the given random variables and that are admissible. Naturally, the fact that the market is complete plays an important role on this possibility of *hedging*. The proof techniques follow closely the ones of theorem 3.5, p.93-94, of [4].

Theorem 1. *Let $x \geq -b(0)$ be given, let $c(\cdot)$ be a consumption process, ξ a nonnegative $\mathcal{F}(T)$-measurable random variable, and $\phi(\cdot)$ a nonnegative, progressively measurable process with*

$$E\left[\int_0^T \bar{H}_0(u)(c(u) + \eta(t)\phi(u))du + \bar{H}_0(T)\xi\right] = x + b(0)$$

then: $\exists \pi, p$ such that $(c, \pi, p) \in \mathcal{A}(x)$ and $X(T) = \xi$ and $Z(t) = \phi(t)$, for all t.

Proof: Define

$$J(t) = \int_0^t \bar{H}_0(u)(c(u) + \eta(u)\phi(u) - i(u))du$$

and the martingale

$$M(t) = E\left[J(T) + \bar{H}_0(T)\xi|\mathcal{F}(t)\right]$$

Using the martingale representation theorem, there exists a progressively measurable, $\mathbb{R}^N$-valued process ψ that satisfies

$$||\psi||_2^2 = \int_0^T ||\psi(u)||^2 du$$

and

$$M(t) = x + \int_0^t \psi'(u)dW(u)$$

Define $X(\cdot)$ by

$$\frac{X(t)}{D(t)S_0(t)} = \frac{1}{Z_0(t)}(M(t) - J(t)) = \frac{1}{Z_0(t)}E\left[\int_t^T \bar{H}_0(u)(c(u)+\eta(u)\phi(u)-i(u))du + \bar{H}_0(T)\xi \middle| \mathcal{F}(t)\right] \tag{2.13}$$

By Itô's lemma we obtain

$$d\left(\frac{X(t)}{D(t)S_0(t)}\right) = -\frac{c(t) + \eta(t)\phi(t) - i(t)}{D(t)S_0(t)}dt + \frac{1}{Z_0(t)}[\psi'(t) + \theta'(t)(M(t) - J(t))]dW_0(t)$$

We define

$$\pi(t) = \frac{1}{\bar{H}_0(t)}(\sigma'(t))^{-1}(\psi(t) + \theta(t)(M(t) - J(t)) \tag{2.14}$$

$$p(t) = \eta(t)(\phi(t) - X(t)) \tag{2.15}$$

Comparing with (2.6), we obtain $X(T) = \frac{D(T)S_0(T)}{Z_0(T)}E(\bar{H}_0(T)\xi|\mathcal{F}(T)) = \xi$ and $Z(t) = \phi(t)$, $t \in [0,T]$ almost surely, and observing that

$$\frac{1}{Z_0(t)}E\left[\int_t^T i(u)\bar{H}_0(u)du\middle|\mathcal{F}(t)\right] = E_0\left[\int_t^T \frac{i(u)}{D(u)S_0(u)}du\middle|\mathcal{F}(t)\right] = \frac{b(t)}{D(t)S_0(t)}$$

we have

$$\frac{X(t)+b(t)}{D(t)S_0(t)} = \frac{1}{Z_0(t)}E\left[\int_t^T \bar{H}_0(u)(c(u)+\eta(u)\phi(u))du\middle|\mathcal{F}(t)\right]$$

and so $X(t) + b(t) \geq 0, \forall t \in [0,T]$. So $(c, \pi, p) \in \mathcal{A}(x)$.

We need only to check that these strategies satisfy the integrability conditions. Observe that $M(\cdot)$ has continuous paths and $||M||_\infty = \max_{0\leq t\leq T}|M(t)| < \infty$, almost surely. Analogously, $||J||_\infty < \infty$, $||S_0||_\infty < \infty$ almost surely and $\kappa_1 = ||1/Z_0(t)||_\infty < \infty$, $\kappa_2 = ||D(t)||_\infty < \infty$ and $\kappa_3 = \int_0^T \eta(u)du < \infty$, almost surely.

$$\begin{aligned}\int_0^T |\pi'(t)(\mu(t)+\delta(t)-r(t)\underline{1}|dt &= \int_0^T \frac{D(t)S_0(t)}{Z_0(t)}|\psi'(t)\theta(t) + ||\theta(t)||^2(M(t)-J(t))|dt \\ &\leq \kappa_1\kappa_2||S_0|||_\infty(||\psi||_2||\theta||_2 + ||\theta||_2^2(||M||_\infty + ||J||_\infty)) < \infty\end{aligned}$$

$$\int_0^T ||\sigma'(t)\pi(t)||^2 dt = \int_0^T \frac{D(t)^2S_0(t)^2}{Z_0(t)^2}\Big|\Big|\psi(t) + \theta(t)(M(t)-J(t))\Big|\Big|^2 dt \leq$$
$$\leq \kappa_1^2\kappa_2^2||S_0||_\infty^2\Big(||\psi|_2^2 + ||\theta|_2^2(||M||_\infty + ||J||_\infty)^2 + 2(||M||_\infty + ||J||_\infty)||\psi||_2||\theta||_2\Big) < \infty$$

$$\begin{aligned}\int_0^T |p(t)|dt &\leq \int_0^T \eta(t)\Big(\phi(t) + \frac{S_0(t)D(t)}{Z_0(t)}|M(t)-J(t)|\Big)dt \leq \\ &\leq \int_0^T \frac{1}{\bar{H}_0(t)}\eta(t)\bar{H}_0(t)\phi(t)dt + ||S_0||_\infty\kappa_1\kappa_2\kappa_3(||M||_\infty + ||J||_\infty) \leq \\ &\leq \kappa_1\kappa_2||S_0||_\infty||J||_\infty + ||S_0||_\infty\kappa_1\kappa_2\kappa_3(||M||_\infty + ||J||_\infty) < \infty\end{aligned}$$

$\square$

2.2.2 Utility maximization

The following lemma is the key tool to restate the above problem as an equivalent one with a fixed planning horizon (see [14] for a proof).

Lemma 1. *If the random variable τ is independent of the filtration $\{\mathcal{F}(t)\}$, then*

$$V(x) = \sup_{(c,\pi,p)\in\mathcal{A}_1(x)} E\left[\int_0^T \bar{F}(s)U_1(s,c(s)) + f(s)U_2(s,Z(s))ds + \bar{F}(T)U_3(X(T))\right]$$

where $\bar{F}(t)$ and $f(t)$ are given by (2.2)

The intuition behind this lemma is that the agent acts as if he will live up to time T, but his time preferences are multiplied by a factor that represents his "force of mortality".

We define the following auxiliary function

$$\mathcal{X}(y) = E\left[\int_0^T \bar{H}_0(t)I_1\left(t,\frac{y\bar{H}_0(t)}{\bar{F}(t)}\right) + \eta(t)\bar{H}_0(t)I_2\left(t,\frac{y\bar{H}_0(t)\eta(t)}{f(t)}\right)dt + \bar{H}_0(T)I_3\left(\frac{y\bar{H}_0(T)}{\bar{F}(T)}\right)\right]$$

Assumption 2.

$$\mathcal{X}(y) < \infty, \forall y > 0$$

Lemma 2. *Under assumptions 2, we have*

i) $\mathcal{X}(\cdot)$ *is continuous, non-increasing on* $(0,\infty)$*;*

ii)

$$\mathcal{X}(0+) = \infty;$$

iii)

$$\mathcal{X}(\infty) = E\left[\int_0^T \bar{H}_0(t)(\bar{c}(t) + \eta(t)\bar{Z}(t))dt + \bar{H}_0(T)\bar{x}\right];$$

iv) $\mathcal{X}(\cdot)$ *is strictly decreasing on* $(0,r)$*, where* $r = \sup\{y > 0 : \mathcal{X}(y) > \mathcal{X}(\infty)\} > 0$;

So $\mathcal{X}(\cdot)$ restricted to $(0,r)$ has an inverse, that is strictly decreasing and denoted by $\mathcal{Y} : (\mathcal{X}(\infty), \infty) \to (0,r)$, i.e., $\mathcal{X}(\mathcal{Y}(x)) = x$ for $x \in (\mathcal{X}(\infty), \infty)$.

Proof: For fixed $t \in [0,T]$, the functions

$$I_1\left(t, \frac{y\bar{H}_0(t)}{\bar{F}(t)}\right), I_2\left(t, \frac{y\bar{H}_0(t)\eta(t)}{f(t)}\right), I_3\left(\frac{y\bar{H}_0(T)}{\bar{F}(T)}\right)$$

are non increasing in y, so is $\mathcal{X}(y)$. Recall that we have $I_1(t, 0+) = I_2(t, 0+) = I_3(0+) = \infty$, and because these functions are non increasing, the monotone convergence theorem implies $\mathcal{X}(0+) = \infty$, so we have ii). By continuity of $I_1(t,\cdot), I_2(t,\cdot)$ and $I_3(\cdot)$ and the fact that they are non increasing, the monotone convergence theorem also implies right continuity of $\mathcal{X}(\cdot)$. Again by continuity of $I_1(t,\cdot), I_2(t,\cdot)$ and $I_3(\cdot)$ and assumption (2), the dominated convergence theorem implies left continuity of $\mathcal{X}(\cdot)$. So $\mathcal{X}(\cdot)$ is a continuous function. Equality in iii) is also implied by the dominated convergence theorem and equalities $\lim_{c\to\infty} I_1(t,c) = \bar{c}(t), \lim_{z\to\infty} I_2(t,z) = \bar{Z}(t)$ and $\lim_{x\to\infty} I_3(x) = \bar{x}$.
To prove assertion iv), let $y \in (0,r)$. So we have $\mathcal{X}(y) > \infty$, and comparing the two expressions and using the observations we made in section 2.1.4, we find that at least one of the following three possibilities occur:

1. $\frac{y\bar{H}_0(t,\omega)}{\bar{F}(t)} < U_1'(t, \bar{c}(t)+)$, for all (t,ω) in a set with positive product measure[2];
2. $\frac{y\bar{H}_0(t,\omega)}{\eta(t)F(t)} < U_2'(t, \bar{Z}(t)+)$, for all (t,ω) in a set with positive product measure;
3. $\frac{y\bar{H}_0(T,\omega)}{\bar{F}(T)} < U_3'(\bar{x}+)$, for all ω in a set with positive P measure.

But we know that $I_1(t,\cdot)$ is *strictly decreasing* on $(0, U_1'(t,\bar{c}(t)+))$, $I_2(t,\cdot)$ is *strictly decreasing* on $(0, U_2'(t,\bar{Z}(t)+))$, and $I_3(\cdot)$ is *strictly decreasing* on $(0, U_3'(\bar{x}+))$. Thus, either one of the above inequalities implies $\mathcal{X}(y-\epsilon) > \mathcal{X}(y)$, for all $\epsilon \in (0,y)$. As this argument applies to any $y \in (0,r)$, we conclude that $\mathcal{X}(\cdot)$ is strictly decreasing on $(0,r)$ and we have iv). □

Remark 3. *Under assumption 1, and the fact that $\eta(t)$ is almost surely uniformly bounded, we have*

$$E\left[\int_0^T \bar{H}_0(t)dt + \bar{H}_0(T)\right] < \infty$$

So

$$\mathcal{X}(\infty) = E\left[\int_0^T \bar{H}_0(t)(\bar{c}(t) + \eta(t)\bar{Z}(t))dt + \bar{H}_0(T)\bar{x}\right] < \infty$$

[2]When we refer to product measure, we mean $Leb_[0,T] \otimes P$, where $Leb_[0,T]$ is the Lebesgue measure on the interval $[0,T]$, and $\otimes$ denotes the product of measures.

From the constraints in remark 2, we obtain

$$E\left[\int_0^T \bar{H}_0(t)(c(t)+\eta(t)Z(t))dt + \bar{H}_0(T)X(T)\right] \geq \mathcal{X}(\infty)$$

In light of the previous remark and the budget constraint, if $x+b(0) \in (0, \mathcal{X}(\infty))$ we have $\mathcal{A}_1(x) = \emptyset$, and $V(x) = -\infty$. For the case when $x + b(0) = \mathcal{X}(\infty)$, actually, any triple $(c, \pi, p) \in \mathcal{A}_1(x)$ must satisfy $c(t) = \bar{c}(t), Z(t) = \bar{Z}(t)$ and $X(T) = \bar{x}$, for Lebesgue almost every $t \in [0, T]$ almost surely, and by Theorem 1, we can find both a portfolio process $\bar{\pi}$ and a premium process $\bar{p}$, that lead to the given bequest and terminal wealth. So we have

$$V(x) = \begin{cases} -\infty, & x+b(0) < \mathcal{X}(\infty) \\ \int_0^T \bar{F}(t)U_1(t,\bar{c}(t)) + f(t)U_2(t,\bar{Z}(t))dt + U_3(\bar{x}) & x+b(0) = \mathcal{X}(\infty) \end{cases}$$

Observe that when $x+b(0) = \mathcal{X}(\infty)$, from remark 1, we have that $V(x) < \infty$, although it may be $-\infty$. So now we need only consider initial wealth x such that $x + b(0)$ is in the domain $(\mathcal{X}(\infty), \infty)$ of $\mathcal{Y}(\cdot)$. We can now state our problem in an equivalent form. For such a x, find:

$$V(x) = \sup_{(c,\pi,p)\in\mathcal{A}_1(x)} E\left[\int_0^T \bar{F}(s)U_1(s,c(s)) + f(s)U_2(s,Z(s))ds + \bar{F}(T)U_3(X(T))\right] \tag{2.16}$$

subject to the *budget constraint*

$$E\left[\bar{H}_0(T)X(T) + \int_0^T \bar{H}_0(t)(c(t)+\eta(t)Z(t))dt\right] \leq x + b(0) \tag{2.17}$$

We now use a "Lagrange multipliers" argument to find the optimal strategies. Let $y > 0$ be a Lagrange multiplier. We restate the previous problem as the unconstrained maximization of

$$E\left[\int_0^T \bar{F}(s)U_1(s,c(s)) + f(s)U_2(s,Z(s))ds + \bar{F}(T)U_3(X(T))\right] +$$
$$+y\left(x+b(0) - E\left[\bar{H}_0(T)X(T) + \int_0^T \bar{H}_0(t)(c(t)+\eta(t)Z(t))dt\right]\right)$$

The previous expression is equal to

$$
\begin{aligned}
&(x+b(0))y + E\left[\int_0^T \bar{F}(t)U_1(t,c(t)) - y\bar{H}_0(t)c(t)dt\right] + \\
&+E\left[\int_0^T f(t)U_2(t,Z(t)) - y\bar{H}_0(t)\eta(t)Z(t)dt\right] + E\Big[\bar{F}(T)U_3(X(T)) - y\bar{H}_0(T)X(T)\Big] \leq \\
&\leq (x+b(0))y + E\left[\int_0^T \bar{F}(t)\widetilde{U}_1\Big(t,\frac{y\bar{H}_0(t)}{\bar{F}(t)}\Big)dt\right] + E\left[\int_0^T f(t)\widetilde{U}_2\Big(t,\frac{y\bar{H}_0(t)\eta(t)}{f(t)}\Big)dt\right] \\
&\qquad +E\left[\bar{F}(T)\widetilde{U}_3\Big(\frac{y\bar{H}_0(T)}{\bar{F}(T)}\Big)dt\right]
\end{aligned}
$$

with equality if and only if

$$
\begin{aligned}
c(t) &= I_1\left(t,\frac{y\bar{H}_0(t)}{\bar{F}(t)}\right)\\
Z(t) &= I_2\left(t,\frac{y\bar{H}_0(t)\eta(t)}{f(t)}\right)\\
X(T) &= I_3\left(\frac{y\bar{H}_0(T)}{\bar{F}(T)}\right)
\end{aligned}
$$

In order to satisfy the budget constraint with equality, we must have $\mathcal{X}(y) = x + b(0)$. As we are considering initial wealth x such that $x + b(0) > \mathcal{X}(\infty)$, then, by lemma 2, there is a unique solution to this equation, which is $y = \mathcal{Y}(x + b(0))$.

So we obtain

$$
\begin{aligned}
c^*(t) &= I_1\left(t,\frac{\mathcal{Y}(x+b(0))\bar{H}_0(t)}{\bar{F}(t)}\right) &(2.18)\\
\Psi^*(t) &= I_2\left(t,\frac{\mathcal{Y}(x+b(0))\bar{H}_0(t)\eta(t)}{f(t)}\right) &(2.19)\\
\xi^* &= I_3\left(\frac{\mathcal{Y}(x+b(0))\bar{H}_0(T)}{\bar{F}(T)}\right) &(2.20)
\end{aligned}
$$

We prove optimality rigorously on the next theorem.

Theorem 2. *Under the previous assumptions, let x be given, such that $x + b(0) \in (\mathcal{X}(\infty), \infty)$, and c^*, Ψ^*, ξ^* be given by (2.18),(2.19),(2.20). Then $\exists (c^*, \pi^*, p^*) \in \mathcal{A}_1(x)$ that is optimal and such that $X^*(T) = \xi^*$ and $Z^*(t) = \Psi^*(t)$, and the value function is*

$$
V(x) = E\left[\int_0^T \bar{F}(t)U_1(t,c^*(t)) + f(t)U_2(t,Z^*(t))dt + \bar{F}(t)U_3(X^*(T))\right]
$$

Proof: By definition of $\mathcal{X}(\cdot)$ and lemma 2, we have

$$E\left[\int_0^T \bar{H}_0(t)c^*(t) + \bar{H}_0(t)\eta(t)Z^*(t)dt + \bar{H}_0(T)\xi^*\right] = \mathcal{X}(\mathcal{Y}(x+b(0))) = x + b(0)$$

So equations (2.19), and (2.20) satisfy the hypothesis of theorem 1 and we obtain $(c^*, \pi^*, p^*) \in \mathcal{A}(x)$ such that $X^*(T) = \xi^*$ and $Z^*(t) = \Psi^*(t)$.

We first prove that $(c^*, \pi^*, p^*) \in \mathcal{A}_1(x)$. Using $\hat{c}$ and $\hat{z}$ as they were defined on remark 1, and the definition of convex duality we obtain

$$\bar{F}(t)U_1(t, c^*(t)) - \mathcal{Y}(x+b(0))\bar{H}_0(t)c^*(t) = \bar{F}(t)\widetilde{U}_1\left(t, \frac{\mathcal{Y}(x+b(0))\bar{H}_0(t)}{\bar{F}(t)}\right) \geq$$
$$\geq \bar{F}(t)\left(U_1(t, \hat{c}+\hat{z}) - \frac{\mathcal{Y}(x+b(0))\bar{H}_0(t)(\hat{c}+\hat{z})}{\bar{F}(t)}\right)$$

$$f(t)U_2(t, Z^*(t)) - \mathcal{Y}(x+b(0))\bar{H}_0(t)\eta(t)Z^*(t) = f(t)\widetilde{U}_2\left(t, \frac{\mathcal{Y}(x+b(0))\bar{H}_0(t)\eta(t)}{f(t)}\right) \geq$$
$$\geq f(t)\left(U_2(t, \hat{c}+\hat{z}) - \frac{\mathcal{Y}(x+b(0))\bar{H}_0(t)\eta(t)(\hat{c}+\hat{z})}{f(t)}\right)$$

$$\bar{F}(T)U_3(X^*(T)) - \mathcal{Y}(x+b(0))\bar{H}_0(T)X^*(T) = \bar{F}(T)\widetilde{U}_3\left(\frac{\mathcal{Y}(x+b(0))\bar{H}_0(T)}{\bar{F}(T)}\right) \geq$$
$$\geq \bar{F}(T)\left(U_3(\hat{c}+\hat{z}) - \frac{\mathcal{Y}(x+b(0))\bar{H}_0(T)(\hat{c}+\hat{z})}{\bar{F}(T)}\right)$$

Using these three inequalities, we obtain

$$E\left(\int_0^T \min\{0, \bar{F}(t)U_1(t, c^*(t))\} + \min\{0, f(t)U_2(t, Z^*(t))\}dt + \min\{0, \bar{F}(T)U_3(X^*(T))\}\right] \geq$$
$$\geq E\left(\int_0^T \min\{0, \bar{F}(t)U_1(t, \hat{c}+\hat{z})\} + \min\{0, f(t)U_2(t, \hat{c}+\hat{z})\}dt + \min\{0, \bar{F}(T)U_3(\hat{c}+\hat{z})\}\right]$$
$$-\mathcal{Y}(x+b(0))(\hat{c}+\hat{z})E\left[\int_0^T \bar{H}_0(t) + \bar{H}_0(t)\eta(t)dt + \bar{H}_0(T)\right] > -\infty$$

So we conclude that $(c^*, \pi^*, p^*) \in \mathcal{A}_1(x)$.

We now prove optimality. For that, let $(c', \pi', p') \in \mathcal{A}_1(x)$, and denote by Z' and $X'(T)$, the legacy and terminal wealth, respectively, induced by (c', π', p'). Again using the definitions of convex duality we obtain

$$\bar{F}(t)U_1(t,c^*(t)) - \mathcal{Y}(x+b(0))\bar{H}_0(t)c^*(t) \geq \bar{F}(t)U_1(t,c'(t)) - \mathcal{Y}(x+b(0))\bar{H}_0(t)c'(t)$$

$$f(t)U_2(t,Z^*(t)) - \mathcal{Y}(x+b(0))\bar{H}_0(t)\eta(t)Z^*(t) \geq f(t)U_2(t,Z'(t)) - \mathcal{Y}(x+b(0))\bar{H}_0(t)\eta(t)Z'(t)$$

$$\bar{F}(T)U_3(X^*(T)) - \mathcal{Y}(x+b(0))\bar{H}_0(T)X^*(T) \geq \bar{F}(T)U_3(X'(T)) - \mathcal{Y}(x+b(0))\bar{H}_0(T)X'(T)$$

So, using the previous inequalities and the budget constraint, we obtain

$$\begin{aligned}
E\left[\int_0^T \bar{F}(t)U_1(t,c^*(t)) + f(t)U_2(t,Z^*(t))dt + \bar{F}(T)U_3(X^*(T))\right] \geq \\
\geq E\left[\int_0^T \bar{F}(t)U_1(t,c'(t)) + f(t)U_2(t,Z'(t))dt + \bar{F}(T)U_3(X'(T))\right] + \\
+\mathcal{Y}(x+b(0))E\left[\int_0^T \bar{H}_0(t)c^*(t) + \bar{H}_0(t)\eta(t)Z^*(t)dt + \bar{H}_0(T)X^*(T)\right] - \\
-\mathcal{Y}(x+b(0))E\left[\int_0^T \bar{H}_0(t)c'(t) + \bar{H}_0(t)\eta(t)Z'(t)dt + \bar{H}_0(T)X'(T)\right] \geq \\
\geq E\left[\int_0^T \bar{F}(t)U_1(t,c'(t)) + f(t)U_2(t,Z'(t))dt + \bar{F}(T)U_3(X'(T))\right]
\end{aligned}$$

So (c^*, π^*, p^*) is optimal. □

Remark 4 (Uniqueness). *Assuming that $V(x) < \infty$, from the proof of the theorem we also obtain that $c^*(\cdot)$, $Z^*(\cdot)$ and $\xi^*(\cdot)$ are* unique *up to almost-everywhere equivalence with respect to the product measure. Note that this implies that both the optimal portfolio $\pi^*(\cdot)$ and the optimal life insurance premium $p^*(\cdot)$ are* unique *up to almost everywhere equivalence with respect to the product measure.*

Corollary 1. *Under the assumptions of theorem 2, the optimal wealth process is given by*

$$X^*(t) = \frac{1}{\bar{H}_0(t)} E\left[\int_t^T \bar{H}_0(u)(c^*(u) + \eta(u)Z^*(u) - i(u))du + \bar{H}_0(T)\xi^* \middle| \mathcal{F}(t)\right] \tag{2.21}$$

The optimal portfolio is given by

$$\pi(t) = \frac{(\sigma'(t))^{-1}\psi(t)}{\bar{H}_0(t)} + (\sigma'(t))^{-1}X^*(t)\theta(t) \tag{2.22}$$

where $\psi(\cdot)$ is the integrand in the stochastic integral representation $M(t) = x+\int_0^t \psi'(t)dW(u)$ of the martingale

$$M(t) = E\left[\int_0^T \bar{H}_0(t)(c^*(u) + \eta(u)Z^*(u) - i(u))du + \bar{H}_0(T)\xi^* \middle| \mathcal{F}(t)\right] \tag{2.23}$$

The optimal life insurance premium is

$$p^*(t) = \eta(t)(Z^*(t) - X^*(t)) \tag{2.24}$$

Proof: These formulas come directly from (2.13,2.14,2.15) by rewriting them using the optimal triple (c^*, π^*, p^*) of the previous theorem. $\square$

Chapter 3

Deterministic Coefficients

In this chapter we restrict ourselves to the case where the coefficients of the model are deterministic functions. The main difference is that with this coefficients, we have that some processes considered before have are Markov processes, and we can explore the connection between these Markovian processes and partial differential equations and deduce more about the form of the optimal strategies obtained before. In particular we derive a mutual fund result similar to the one obtained by Merton in ([6, 7]). We then derive the *Hamilton-Jacobi-Bellman* associated with the optimal control problem stated before, which is a second-order *nonlinear* partial differential equation satisfied by the value function. We conclude by considering utilities that have constant relative risk aversion, and on that case we are able to compute the solution in closed form.

Throughout this chapter, we will always assume that $A(\cdot) = 0$. We will assume that $r(\cdot), \sigma(\cdot), \theta(\cdot)$ and $\eta(\cdot)$ are *continuous and deterministic* functions, and that $i(\cdot)$ is *deterministic* and satisfies the integrability condition imposed in section 2. So we have $S_0(t) = \exp\{-\int_0^t r(u)du\}$, and $b(t)$ is also deterministic and simplifies to

$$b(t) = \int_t^T \frac{i(s)}{\exp\{\int_t^s r(u) + \eta(u)du\}} ds$$

3.1 Mutual fund theorem

We will assume that $||\theta(\cdot)||$ is bounded away from zero and that $r(\cdot)$, $\eta(\cdot)$, $\lambda(\cdot)$ and $||\theta(\cdot)||$ are Hölder continuous[1], i.e., for some $K > 0$ and $\rho \in (0, 1)$ we have, for all $t_1, t_2 \in [0, T]$

[1]So, by boundedness, the Novikov condition mentioned on chapter 1 implies that $Z_0(\cdot)$ is indeed a martingale

$$|r(t_1) - r(t_2)| \le K|t_1 - t_2|^\rho, \quad \Big| ||\theta(t_1)|| - ||\theta(t_2)|| \Big| \le K|t_1 - t_2|^\rho$$

$$|\eta(t_1) - \eta(t_2)| \le K|t_1 - t_2|^\rho, \quad |\lambda(t_1) - \lambda(t_2)| \le K|t_1 - t_2|^\rho$$

We shall represent the optimal consumption, the optimal bequest and the optimal portfolio in "feedback form" on the level of optimal (full) wealth at time t, $X^*(t)+b(t)$, i.e.,

$$c(t) = C(t, X^*(t) + b(t)), \quad Z(t) = Z(t, X^*(t) + b(t)), \quad \pi(t) = \Pi(t, X^*(t) + b(t))$$

for functions C, Z, Π that do not depend on the initial wealth. By theorem (1), we have the optimal life insurance premium in feedback form

$$p(t) = \eta(t) \left(Z(t, X^*(t) + b(t)) - X^*(t) \right)$$

We will impose the following conditions on the agent's preference structure

Assumption 3. *The agent's preference structure* (U_1, U_2, U_3) *satisfies*

i) (polynomial growth of I_1, I_2 *and* I_3*) there is a constant* $\gamma > 0$ *such that*

$$I_1(t,y) \le \gamma + y^{-\gamma}, \forall (t,y) \in [0,T] \times (0,\infty),$$

$$I_2(t,y) \le \gamma + y^{-\gamma}, \forall (t,y) \in [0,T] \times (0,\infty),$$
$$I_3(y) \le \gamma + y^{-\gamma}, \forall y \in (0,\infty);$$

ii) (polynomial growth of $|U_1 \circ I_1|, |U_2 \circ I_2|$ *and* $|U_3 \circ I_3|$*) there is a constant* $\gamma > 0$ *such that*

$$|U_1(t, I_1(t,y))| \le \gamma + y^\gamma + y^{-\gamma}, \forall (t,y) \in [0,T] \times (0,\infty),$$

$$|U_2(t, I_2(t,y))| \le \gamma + y^\gamma + y^{-\gamma}, \forall (t,y) \in [0,T] \times (0,\infty),$$
$$|U_3(I_3(y))| \le \gamma y^\gamma + y^{-\gamma}, \forall y \in (0,\infty);$$

iii) (Hölder continuity of I_1 *and* I_2*) for each* $y_0 \in (0,\infty)$, *there exist constants* $\varepsilon(y_0) > 0, K(y_0) > 0$, *and* $\rho(y_0) \in (0,1)$, *such that*

$$|I_1(t,y) - I_1(t,y_0)| \le K(y_0)|y - y_0|^{\rho(y_0)}, \forall (t,y) \in [0,T] \times (0,\infty) \cap B(y_0; \varepsilon(y_0))^2,$$

$$|I_2(t,y) - I_2(t,y_0)| \le K(y_0)|y - y_0|^{\rho(y_0)}, \forall (t,y) \in [0,T] \times (0,\infty) \cap B(y_0; \varepsilon(y_0));$$

[2] We use the notation: $B(y_0; \varepsilon) = (y_0 - \varepsilon, y_0 + \varepsilon)$

iv) at least one of the following occurs

for each $t \in [0,T]$, $\frac{\partial}{\partial y} I_1(t,y)$ is defined and strictly negative for all y in a set of positive Lebesgue measure,

for each $t \in [0,T]$, $\frac{\partial}{\partial y} I_2(t,y)$ is defined and strictly negative for all y in a set of positive Lebesgue measure,

$\frac{\partial}{\partial y} I_3(y)$ is defined and strictly negative for all y in a set of positive Lebesgue measure;

Remark 5. *We will also need $U_1 \circ I_1, U_2 \circ I_2$ to be Hölder continuous. Indeed, this is implied by the previous assumptions. For $y_0 \in (0,\infty)$ and $\varepsilon(y_0), K(y_0)$ and $\rho(y_0)$ as in the previous assumption, for $y \in (0,\infty) \cap B(y_0, \varepsilon(y_0))$, by the mean value theorem, we have*

$$|U_1(t, I_1(t,y)) - U_1(t, I_1(t,y_0))| \leq U_1'(t, \iota(t))|I_1(t,y) - I_1(t,y_0)| \leq MK(y_0)|y-y_0|^{\rho(y_0}$$

where $\iota(t)$ takes values between $I_1(t,y)$ and $I_1(t,y_0)$ and M is a bound on the continuous function $U_1'(t, I_1(t,v))$ for $(t,v) \in [0,T] \times B(y_0; \varepsilon(y_0))$. This means that $U_1 \circ I_1$ is Hölder continuous, and has the same type of Hölder continuity as imposed for I_1 in assumption 3,iii). The same argument applies to the function $U_2 \circ I_2$.

We define the process $Y^{(t,y)}(s)$, for $t \leq s \leq T$, for given $(t,y) \in [0,T] \times (0,\infty)$.

$$Y^{(t,y)}(s) = y \exp\left\{ -\int_t^s r(u) + \eta(u) - \frac{1}{2}||\theta(u)||^2 - \lambda(u) du - \int_t^s \theta'(u) dW_0(u) \right\} \quad (3.1)$$

with dynamics given by:

$$dY^{(t,y)}(s) = Y^{(t,y)}(s) \left(-r(s)ds - \eta(s)ds + ||\theta(s)||^2 ds - \theta'(s)dW_0(s) \right) \quad (3.2)$$

Observe that $Y^{(t,y)}(t) = y$, $Y^{(t,y)}(s) = yY^{(t,1)}(s) = y\frac{\bar{H}_0(s)}{\bar{H}_0(t)}\frac{\bar{F}(t)}{\bar{F}(s)}$. We will write $\bar{F}(s,t) = \bar{F}(s)/\bar{F}(t)$, which is the conditional probability distribution for the agent to be alive at time s condition upon being alive at time $t \leq s$. We also have that $f(s,t)$, the conditional probability density for the wage earner's death to occur at time s conditional upon being alive at time $t \leq s$ is given by $f(s,t) = \lambda(s)\bar{F}(s,t)$. So $Y^{(t,y)}(s) = y\frac{\bar{H}_0(s)}{\bar{H}_0(t)}\frac{1}{\bar{F}(s,t)}$.

Let us denote $z = \mathcal{Y}(x + b(0))$. We rewrite the process $X^*(t) + b(t)$ from formula (2.21) using the process $Y^{(t,y)}(\cdot)$. By first switching to expectation with respect to P_0 by Bayes's rule, we obtain

$$
\begin{aligned}
&X^*(t) + b(t) = \\
&= E_0 \Bigg[\int_t^T \exp\left(- \int_t^s r(u) + \eta(u) du \right) \left(I_1\big(s, Y^{(0,z)}(s)\big) + \eta(s) I_2\Big(s, \frac{\eta(s)}{\lambda(s)} Y^{(0,z)}(s) \Big) \right) ds \\
&+ \exp\left(- \int_t^T r(u) + \eta(u) du \right) I_3\left(Y^{(0,z)}(T) \right) \Big| \mathcal{F}(t) \Bigg]
\end{aligned}
$$

For $(t, y) \in [0, T] \times (0, \infty)$, we define the function

$$
\begin{aligned}
\mathcal{X}_1(t, y) = E_0 \Bigg[\int_t^T \exp\left(- \int_t^s r(u) + \eta(u) du \right) \left(I_1\big(s, Y^{(t,y)}(s)\big) + \eta(s) I_2\Big(s, \frac{\eta(s)}{\lambda(s)} Y^{(t,y)}(s) \Big) \right) ds \\
+ \exp\left(- \int_t^T r(u) + \eta(u) du \right) I_3\left(Y^{(t,y)}(T) \right) \Bigg]
\end{aligned} \tag{3.3}
$$

Using the Markov property for $Y^{(t,y)}(\cdot)$ under the measure P_0, we have

$$
X^*(t) + b(t) = \mathcal{X}_1(t, Y^{(0,z)}(t))
$$

So we condition $\mathcal{X}_1(t, y)$ upon $\mathcal{F}(t)$ and use the rule of iterated expectations, and apply Bayes's rule two times, and obtain

$$
\begin{aligned}
\mathcal{X}_1(t, y) = E \Bigg\{ \frac{Z_0(T)}{Z_0(t)} E \Bigg[\int_t^T \exp\left(- \int_t^s r(u) + \eta(u) du \right) Z_0(s) I_1\big(s, Y^{(t,y)}(s)\big) \\
+ \exp\left(- \int_t^s r(u) + \eta(u) du \right) Z_0(s) \eta(s) I_2\Big(s, \frac{\eta(s)}{\lambda(s)} Y^{(t,y)}(s) \Big) ds \\
+ \exp\left(- \int_t^T r(u) + \eta(u) du \right) Z_0(T) I_3(Y^{(t,y)}(T)) \Big| \mathcal{F}(t) \Bigg] \Bigg\}
\end{aligned}
$$

If we take $Z_0(t)$ inside the second expectation, and multiply and divide by $\bar{F}(s, t)$ we have

$$
\begin{aligned}
\mathcal{X}_1(t, y) = E \Bigg\{ Z_0(T) E \Bigg[\int_t^T \bar{F}(s, t) \exp\left(- \int_t^s r(u) + \eta(u) - \lambda(u) du \right) \frac{Z_0(s)}{Z_0(t)} I_1\big(s, Y^{(t,y)}(s)\big) + \\
+ \bar{F}(s, t) \exp\left(- \int_t^s r(u) + \eta(u) - \lambda(u) du \right) \frac{Z_0(s)}{Z_0(t)} \eta(s) I_2\Big(s, \frac{\eta(s)}{\lambda(s)} Y^{(t,y)}(s) \Big) ds + \\
+ \bar{F}(T, t) \exp\left(- \int_t^T r(u) + \eta(u) - \lambda(u) du \right) \frac{Z_0(T)}{Z_0(t)} I_3(Y^{(t,y)}(T)) \Big| \mathcal{F}(t) \Bigg] \Bigg\}
\end{aligned}
$$

So we may write

$$\mathcal{X}_1(t,y) = \frac{1}{y}E\Bigg\{Z_0(T)E\Bigg[\int_t^T \bar{F}(s,t)Y^{(t,y)}(s)I_1\big(s,Y^{(t,y)}(s)\big) + \\ +\bar{F}(s,t)Y^{(t,y)}(s)\eta(s)I_2\bigg(s,\frac{\eta(s)}{\lambda(s)}Y^{(t,y)}(s)\bigg)ds + \\ +\bar{F}(T,t)Y^{(t,y)}(T)I_3(Y^{(t,y)}(T))\Big|\mathcal{F}(t)\Bigg]\Bigg\} \tag{3.4}$$

On the other hand, by the Markov property for $Y^{(t,y)}(\cdot)$ under P, we have that

$$E\left[\int_t^T \bar{F}(s,t)Y^{(t,y)}(s)I_1\big(s,Y^{(t,y)}(s)\big) + \bar{F}(s,t)Y^{(t,y)}(s)\eta(s)I_2\bigg(s,\frac{\eta(s)}{\lambda(s)}Y^{(t,y)}(s)\bigg)ds\right]. \\ +\bar{F}(T,t)Y^{(t,y)}(T)I_3\left(Y^{(t,y)}(T)\right)\Big|\mathcal{F}(t)\Big]$$

is a deterministic function (of $Y^{(t,y)}(t) = y$).

So from (3.4), we obtain

$$\mathcal{X}_1(t,y) = \frac{1}{y}E\Bigg[\int_t^T \bar{F}(s,t)Y^{(t,y)}(s)I_1\big(s,Y^{(t,y)}(s)\big) + \\ +\bar{F}(s,t)Y^{(t,y)}(s)\eta(s)I_2\bigg(s,\frac{\eta(s)}{\lambda(s)}Y^{(t,y)}(s)\bigg)ds + \\ +\bar{F}(T,t)Y^{(t,y)}(T)I_3(Y^{(t,y)}(T))\Bigg] \tag{3.5}$$

So $\mathcal{X}_1(t,y)$ provides an extension for the function $\mathcal{X}(\cdot)$ that we have defined on the previous chapter, namely, $\mathcal{X}_1(0,y) = \mathcal{X}(y)$. We will suppress the subscript from now on. Using the same arguments of the proof of lemma 2, we have, for fixed $t \in [0,T)$, $\mathcal{X}(t,0+) = \infty$, and

$$\mathcal{X}(t,\infty) = \int_t^T \exp\left(-\int_t^s r(u)+\eta(u)du\right)(\bar{c}(s)+\eta(s)\bar{Z}(s))ds + \exp\left(-\int_t^T r(u)+\eta(u)du\right)\bar{x}$$

and $\mathcal{X}(t,\cdot)$ is strictly decreasing on $(\mathcal{X}(t,\infty),\infty)$, and thus, has a strictly decreasing inverse function $\mathcal{Y}(t,\cdot)$, that is

$$\mathcal{X}(t,\mathcal{Y}(t,x)) = x, \quad \forall x \in (\mathcal{X}(t,\infty),\infty)$$

Also $\mathcal{X}(T,\cdot) = I_3(\cdot)$, which is strictly decreasing on $(0, U_3'(\bar{x}+))$ and $\mathcal{X}(T,\infty) = \bar{x}$. The inverse of $\mathcal{X}(T,\cdot)$ is $\mathcal{Y}(T,\cdot) = U_3'(\cdot)$ and we also have $\mathcal{X}(T, \mathcal{Y}(T,x)) = x$. We have the following lemma

Lemma 3. *Under the previous assumptions, the function $\mathcal{X}$ is of class $C([0,T] \times (0,\infty)) \cap C^{1,2}([0,T) \times (0,\infty))$ and solves the Cauchy problem*

$$
\begin{aligned}
&\mathcal{X}_t(t,y) + \frac{1}{2}||\theta(t)||^2 y^2 \mathcal{X}_{yy}(t,y) + \left(||\theta(t)||^2 - r(t) - \eta(t) + \lambda(t)\right) y \mathcal{X}_y(t,y) \\
&- (r(t) + \eta(t))\, \mathcal{X}(t,y) \\
&= -I_1(t,y) - \eta(t) I_2\left(t, \frac{\eta(t)}{\lambda(t)} y\right), \qquad (t,y) \in [0,T) \times (0,\infty) \qquad (3.6) \\
&\mathcal{X}(T,y) = I_3(y), \qquad y \in (0,\infty) \qquad (3.7)
\end{aligned}
$$

Furthermore, the inverse function $\mathcal{Y}$ is of class $C^{1,2}$ on $\{(t,x) \in [0,T) \times \mathbb{R}, x > \mathcal{X}(t,\infty)\}$ and is continuous on $\{(t,x) \in [0,T] \times \mathbb{R}, x > \mathcal{X}(t,\infty)\}$

Proof: Consider the Cauchy problem:

$$
\begin{aligned}
&u_t(t,v) + \frac{1}{2}||\theta(t)||^2 u_{vv}(t,v) + \left(\frac{1}{2}||\theta(t)||^2 - r(t) - \eta(t) + \lambda(t)\right) u_v(t,v) - (r(t) + \eta(t))\, u(t,v) \\
&= -I_1(t, e^v) - \eta(t) I_2\left(t, \frac{\eta(t)}{\lambda(t)} e^v\right), \qquad (t,v) \in [0,T) \times \mathbb{R} \qquad (3.8) \\
&u(T,v) = I_3(e^v), \qquad v \in \mathbb{R} \qquad (3.9)
\end{aligned}
$$

From the theory of partial differential equations, (see [2], section 1.7), there is a solution u of class $C([0,T] \times \mathbb{R}) \cap C^{1,2}([0,T) \times \mathbb{R})$, and such that, for $\varepsilon > 0$, there is a constant $C(\varepsilon)$ such that $|u(t,v)| \leq C(\varepsilon) e^{\varepsilon v^2}$, for all $v \in \mathbb{R}$.

We fix $(t,y) \in [0,T) \times (0,\infty)$. From (3.2), by Itô's rule, we obtain

$$
d \log Y^{(t,y)}(s) = \left(-r(s)ds - \eta(s)ds + \lambda(s)ds + \frac{1}{2}||\theta(s)||^2 ds - \theta'(s) dW_0(s)\right) \qquad (3.10)
$$

Again by Itô's rule, and using (3.8) we compute

$$
\begin{aligned}
&d\left[\exp\left(-\int_t^s r(u) + \eta(u) du\right) u(s, \log Y^{(t,y)}(s))\right] = \\
&- \exp\left(-\int_t^s r(u) + \eta(u) du\right) I_1\left(s, Y^{(t,y)}(s)\right) ds \\
&- \exp\left(-\int_t^s r(u) + \eta(u) du\right) \eta(s) I_2\left(s, \frac{\eta(s)}{\lambda(s)} Y^{(t,y)}(s)\right) ds \\
&- \exp\left(-\int_t^s r(u) + \eta(u) du\right) u_v(s, \log Y^{(t,y)}(s)) \theta'(s) dW_0(s) \qquad (3.11)
\end{aligned}
$$

For $n \in \mathbb{N}$, define $\tau_n = (T - 1/n) \wedge \inf\{s \in [t,T]; |\log Y^{(t,y)}(s)| \geq n\}$, and so we have that $\max_{0\leq s \leq \tau_n} u_v(s, \log Y^{(t,y)}(s))|$ is bounded uniformly in $\omega \in \Omega$. We integrate (3.11) from t to τ_n, take expectations with respect to P_0. The stochastic term disappears and we obtain

$$\begin{aligned} u(t, \log y) &= E_0 \Bigg[\int_t^{\tau_n} \exp\Big(-\int_t^s r(u) + \eta(u) du \Big) I_1\Big(s, Y^{(t,y)}(s) \Big) ds + \\ &+ \exp\Big(-\int_t^s r(u) + \eta(u) du \Big) \eta(s) I_2\Big(s, \frac{\eta(s)}{\lambda(s)} Y^{(t,y)}(s) \Big) \Bigg] \\ &+ E_0 \left[\exp\Big(-\int_t^{\tau_n} r(u) + \eta(u) du \Big) u(\tau_n, \log Y^{(t,y)}(\tau_n)) \right]. \end{aligned} \tag{3.12}$$

We now make n go to infinity. By the monotone convergence theorem, the limit of the first term is

$$E_0 \left[\int_t^T \exp\Big(-\int_t^s r(u) + \eta(u) du \Big) \Big(I_1\big(s, Y^{(t,y)}(s)\big) ds + \eta(s) I_2 \Big(s, \frac{\eta(s)}{\lambda(s)} Y^{(t,y)}(s) \Big) \Big) \right]$$

For the second term we want to use the dominated convergence theorem, so we need to find a dominating function. Using the growth condition exposed before, we may write

$$\begin{aligned} &\left| \exp\Big(-\int_t^s r(u) + \eta(u) du \Big) u\left(\tau_n, \log Y^{(t,y)}(\tau_n)\right) \right| \\ &\leq C(\varepsilon) \exp\Big(\int_t^s |r(u)| + |\eta(u)| du \Big) \exp\left(\varepsilon (\log Y^{(t,y)}(\tau_n))^2 \right) \\ &\leq C(\varepsilon) \exp\Big(\int_t^s |r(u)| + |\eta(u)| du \Big) \exp\Bigg(\varepsilon \Bigg[|\log y| + \int_t^T \Big| \frac{1}{2} ||\theta(u)||^2 - r(u) - \eta(u) + \lambda(u) \Big| du \\ &+ \sup_{t \leq s \leq T} \Big| \int_t^s \theta'(u) dW_0(u) \Big| \Bigg]^2 \Bigg) \\ &\leq C(\varepsilon) \exp\Big(\int_t^s |r(u)| + |\eta(u)| du \Big) \exp\Bigg(2\varepsilon \left[|\log y| + \int_t^T \Big| \frac{1}{2} ||\theta(u)||^2 - r(u) - \eta(u) + \lambda(u) \Big| du \right]^2 \\ &+ 2\varepsilon \Big(\sup_{t \leq s \leq T} \Big| \int_t^s \theta'(u) dW_0(u) \Big| \Big)^2 \Bigg) \end{aligned} \tag{3.13}$$

It can be shown (see [4], pp.124-125) that, for a convenient choice of ε, we have

$$E_0\left[\exp\left(\varepsilon\left(\sup_{t\le s\le T}\Big|\int_t^s \theta'(u)dW_0(u)\Big|\right)^2\right)\right] < \infty \tag{3.14}$$

So we may use the dominated convergence theorem to find the limit of the second term in (3.12), which is

$$E_0\left[\exp\left(-\int_t^s r(u)+\eta(u)du\right) I_3(Y^{(t,y)}(T))\right]$$

So we have

$$\mathcal{X}(t,y) = u(t,\log y), (t,y) \in [0,T]\times(0,\infty)$$

and we have that $\mathcal{X}$ is of class $C([0,T]\times(0,\infty))\cap C^{1,2}([0,T)\times(0,\infty))$. We can compute the derivatives of u using the derivatives of $\mathcal{X}$ by the chain rule

$$u_t(t,\log y) = \mathcal{X}_t(t,y)$$

$$u_y(t,\log y) = y\mathcal{X}_y(t,y)$$

$$u_{yy}(t,\log y) = y^2\mathcal{X}_{yy}(t,y) + u_y(t,\log y) = y^2\mathcal{X}_{yy}(t,y) + y\mathcal{X}_y(t,y)$$

After substitution on (3.8,3.9), we obtain (3.6, 3.7).

We now need only to check differentiability of the inverse $\mathcal{Y}$. We have $\int_t^T \theta'(u)dW_0(u)$ has normal distribution with mean zero and variance $\rho^2 = \int_t^T ||\theta(u)||^2du$. We now use assumption 3, iv) and assume that $I_3'(y)$ is defined and strictly negative[3] on a set N with positive Lebesgue measure. $I_1(t,\cdot)$ and $I_2(t,\cdot)$ are non increasing, so letting $t\in[0,T), y>0$, and $h>0$, we have

$$\begin{aligned}&\frac{1}{h}[\mathcal{X}(t,y)-\mathcal{X}(t,y+h)]\\ &\ge \exp\left(-\int_t^T r(u)+\eta(u)du\right)E_0\left(\frac{1}{h}[I_3(yY^{(t,1)}(T) - I_3((y+h)Y^{(t,1)}(T)]\right)\end{aligned} \tag{3.15}$$

If we define $m = \int_t^T -r(u)-\eta(u)+(1/2)||\theta(u)||^2+\lambda(u)du$, we can write $Y^{(t,1)}(T) = e^{m-\int_t^T \theta'(u)dW_0(u)}$. So we have

[3]We use assumption 3, iv) applied to the function I_3 just for simplicity of notation. The same argument would apply to I_1 or I_2

$$
\begin{aligned}
&\frac{1}{h}[\mathcal{X}(t,y)-\mathcal{X}(t,y+h)] \\
&\geq \exp\left(-\int_t^T r(u)+\eta(u)du\right)\frac{1}{\sqrt{2\pi}}\int_N \frac{1}{h}\left(I_3\left(ye^{m-\rho\omega}\right)-I_3\left((y+h)e^{m-\rho\omega}\right)\right)e^{-\omega^2/2}d\omega
\end{aligned}
$$

So we let h go to zero, and Fatou's lemma implies, together with the chain rule, that

$$
-\mathcal{X}_y(t,y) \geq -\frac{1}{\sqrt{2\pi}}\int_N e^{m-\rho\omega} I_3'\left(ye^{m-\rho\omega}\right)e^{-\omega^2/2} > 0
$$

So the inverse function theorem implies that the inverse $\mathcal{Y}$ exists, and that it is of class $C^{1,2}$ on $\{(t,x)\in[0,T)\times\mathbb{R}, x>\mathcal{X}(t,\infty)\}$, and it is continuous on $\{(t,x)\in[0,])\times\mathbb{R}, x>\mathcal{X}(t,\infty)\}$, and the proof is complete. □

Using the fact that $\mathcal{X}(t,y)=u(t,\log y),(t,y)\in[0,T]\times(0,\infty)$, we can substitute in (3.11), and after setting $t=0$ and integrating, we obtain the following integral formula

$$
\begin{aligned}
&\exp\left(-\int_0^s r(u)+\eta(u)du\right)\mathcal{X}(s,Y^{(0,y)}(s)) \\
&+\int_0^t \exp\left(-\int_0^s r(u)+\eta(u)du\right)\left(I_1\big(s,Y^{(t,y)}(s)\big)+\eta(s)I_2\big(s,\frac{\eta(s)}{\lambda(s)}Y^{(t,y)}(s)\big)\right)ds \\
&=\mathcal{X}(0,y)-\int_0^t \exp\left(-\int_0^s r(u)+\eta(u)du\right)Y^{(0,y)}(s)\mathcal{X}_y\left(s,Y^{(0,y)}(s)\right)\theta'(s)dW_0(s)
\end{aligned} \tag{3.16}
$$

As we noted on the previous chapter, if $x+b(0)<\mathcal{X}(0,\infty)$, then the expected utility is $-\infty$. If $x+b(0)=\mathcal{X}(0,\infty)$, we must have $c(t)=\bar{c}(t), Z(t)=\bar{Z}(t), X(T)=\bar{x}$, and this gives us the expected utility

$$
\int_0^T \bar{F}(s,t)U_1(s,\bar{c}(s))+f(s,t)U_2(s,\bar{Z}(s))ds+\bar{F}(T,t)U_3(\bar{x}).
$$

Observe that choosing the portfolio to be $\pi=0$, we have the desired terminal wealth. Indeed, substituting this on (2.11), one obtains

$$
\begin{aligned}
&X(t)+b(t)=-\int_0^t \exp\left(-\int_t^s r(u)+\eta(u)du\right)\left(\bar{c}(s)+\eta(s)\bar{Z}(s)\right)ds \\
&+\exp\left(\int_0^t r(u)+\eta(u)du\right)\mathcal{X}(0,\infty)=\mathcal{X}(t,\infty)
\end{aligned}
$$

where we have used the formulas for $\mathcal{X}(0,\infty)$ and $\mathcal{X}(t,\infty)$. For $t=T$ we have $X(T)=\bar{x}$. So the total wealth at time t is $\mathcal{X}(t,\infty)$ and we have the feedback form for optimal consumption, bequest and investment

$$C(t,\mathcal{X}(t,\infty))=\bar{c}(t),\quad Z(t,\mathcal{X}(t,\infty))=\bar{Z}(t)$$
$$\Pi(t,\mathcal{X}(t,\infty))=0$$

We now analyse the last case where the total wealth at time t exceeds $\mathcal{X}(t,\infty)$. If $x+b(0)>\mathcal{X}(0,\infty)$, denoting $z=\mathcal{Y}(0,\infty)$ as before, then we have

$$X^*(t)+b(t)=\mathcal{X}(t,Y^{(0,z)}(t)).$$

As we are assuming that $X^*(t)+b(t)>\mathcal{X}(t,\infty)$, then we have

$$Y^{(0,z)}(t)=\mathcal{Y}(t,X^*(t)+b(t))$$

So we can write the optimal consumption and bequest (2.18), and (2.19)

$$c^*(t)=I_1\left(t,zY^{(0,1)}(t)\right)=I_1\left(t,\mathcal{Y}(t,X^*(t)+b(t))\right) \tag{3.17}$$

$$Z^*(t)=I_2\left(t,\frac{\eta(s)}{\lambda(s)}zY^{(0,1)}(t)\right)=I_2\left(t,\frac{\eta(s)}{\lambda(s)}\mathcal{Y}(t,X^*(t)+b(t))\right) \tag{3.18}$$

Recall that we have $\mathcal{X}(x,\mathcal{Y}(t,x))=x$, for $x>\mathcal{X}(t,\infty)$, so we have

$$\mathcal{X}_y(t,\mathcal{Y}(t,x))=1/\mathcal{Y}_x(t,x).$$

We now use the integral formula (3.16), by rewriting it in terms of $X^*(t)+b(t)$ and for $y=z$ and using this last identity and obtain

$$\begin{aligned}&\frac{X^*(t)+b(t)}{D(t)S_0(t)}+\int_0^t\frac{c^*(s)+\eta(s)Z^*(s)}{D(s)S_0(s)}ds\\&=x+b(0)-\int_0^t\frac{\mathcal{Y}(s,X^*(s)+b(s))/\mathcal{Y}_x(s,X^*(s)+b(s))}{D(s)S_0(s)}\theta'(s)dW_0(s)\end{aligned}$$

By comparing this last equation with (2.11), we conclude that the optimal portfolio is of the following form

$$\pi(t)=-(\sigma'(t))^{-1}\theta(t)\frac{\mathcal{Y}(t,X^*(t)+b(t))}{\mathcal{Y}_x(t,X^*(t)+b(t))} \tag{3.19}$$

The form of the optimal portfolio that we have just derived is generally known as *Merton's mutual fund theorem*, first derived by Merton in [6, 7], without considering a random time horizon. This last expression tells us that to achieve optimality, the agent should always invest according to the proportion

$$(\sigma'(t))^{-1}\theta(t) = (\sigma(t)\sigma'(t))^{-1}\left(\mu(t) + \delta(t) - r(t)\underline{1}\right)$$

that depends only on the characteristics of the market: the volatilities, the mean rate of return, the risk free rate and the dividend rate. This proportion is independent of the utilities that are considered and the wealth levels. So the agent is indifferent about investing in the money market and in the assets individually, or just investing in the money market and in this *mutual fund* which is a linear combination of the available assets. The only thing that actually depends on his utilities and wealth is the quotient $\frac{\mathcal{Y}(t,X^*(t)+b(t))}{\mathcal{Y}_x(t,X^*(t)+b(t))}$ which is the dollar amount he invests on the *mutual fund*.

3.2 Hamilton-Jacobi-Bellman equation

We now proceed to derive the *Hamilton-Jacobi-Bellman* equation associated with our optimal control problem. For that, we must consider a family of stochastic optimal control problems that are also parametrized by the time variable t. For given $(t,x) \in [0,T] \times \mathbb{R}$, we have, for given strategies (c,π,p), and initial condition (t,x) the wealth equation for $t \le s \le T$ is

$$\begin{aligned}
&\exp\left(-\int_t^s r(u) + \eta(u)du\right)(X(s) + b(s)) \\
&+ \int_t^s \exp\left(-\int_t^u r(v) + \eta(v)dv\right)(c(u) + \eta(u)Z(u)du)\,du \\
&= x + b(t) + \int_t^s \exp\left(-\int_t^u r(v) + \eta(v)dv\right)\pi'(u)\sigma(u)dW_0(u)
\end{aligned}$$

We define $\mathcal{A}(t,x)$ to be the set of triples (c,π,p) such that $Z(s) \ge 0$ and $X(s)+b(s) \ge 0$ for all $s \in [t,T]$. And we define the value function as in (2.9), and use lemma (1), and thus we have

$$V(t,x) = \sup_{(c,\pi,p)\in\mathcal{A}_1(t,x)} E\left[\int_t^T \bar{F}(s,t)U_1(s,c(s)) + f(s,t)U_2(s,Z(s))ds + \bar{F}(T,t)U_3(\bar{X}(T))\right]$$

where $\bar{F}(s,t)$ and is $f(s,t)$ are defined as on the previous section and

$$\mathcal{A}_1(t,x) = \left\{ (c,\pi,p) \in \mathcal{A}(t,x) : E\left(\int_t^T \min\{0, \bar{F}(s,t)U_1(s,c(s))\} \right.\right.$$
$$\left.\left. + \min\{0, f(s,t)U_2(s,Z(s))\}ds + \min\{0, \bar{F}(T,t)U_3(X(T))\} \right) > -\infty \right\} \tag{3.20}$$

We define the function

$$\begin{aligned} G(t,y) &= E\left[\int_t^T \bar{F}(s,t)U_1\left(s, I_1\left(s, yY^{(t,1)}(s)\right)\right) \right. \\ &+ f(s,t)U_2\left(s, I_2\left(s, \frac{\eta(s)}{\lambda(s)} yY^{(t,1)}(s)\right)\right) ds \\ &+ \left. \bar{F}(T,t)U_3\left(I_3\left(yY^{(t,1)}(s)\right)\right) \right] \end{aligned} \tag{3.21}$$

Similarly to what we have done on the previous chapter, we have that for $x + b(t) < \mathcal{X}(t,\infty)$, then $V(t,x) = -\infty$, and if $x + b(t) = X(t,\infty)$, then $V(t,x) = \int_t^T U_1(s,\bar{c}(s)) + U_2(s,\bar{Z}(s))ds + U_3(\bar{x})$, and if $x + b(t) > \mathcal{X}(t,\infty)$, then

$$V(t,x) = G(t, \mathcal{Y}(t, x + b(t))) \tag{3.22}$$

And we also have $V(T,x) = U_3(x)$ for $x \in \mathbb{R}$.

We want to be able to compute derivatives of this function G we have just defined, and consequently compute the derivatives of V and establish a Cauchy problem for V. We have the following lemma.

Lemma 4. *Under the previous assumptions, the function G is of class $C([0,T] \times (0,\infty)) \cap C^{1,2}([0,T) \times (0,\infty))$ and solves the Cauchy problem*

$$G_t(t,y) + \frac{1}{2}||\theta(t)||^2 y^2 G_{yy}(t,y) - (r(t) + \eta(t) - \lambda(t))\, yG_y(t,y) - \lambda(t)G(t,y)$$
$$= -U_1(t, I_1(t,y)) - \lambda(t)U_2\left(I_2\left(t, \frac{\eta(t)}{\lambda(t)}y\right)\right), \quad (t,y) \in [0,T) \times (0,\infty) \tag{3.23}$$
$$G(T,y) = U_3(I_3(y)), \qquad y \in (0,\infty) \tag{3.24}$$

Proof: The proof is similar to that of lemma 3. Let u be a solution to the Cauchy problem

$$
\begin{aligned}
&u_t(t,v) + \frac{1}{2}||\theta(t)||^2 u_v(t,v) - \left(r(t) + \eta(t) - \lambda(t)) + \frac{1}{2}||\theta(t)||^2 \right) u_v(t,v) - \lambda(t)u(t,v) \\
&= -U_1\left(t, I_1\left(t, e^v\right)\right) - \lambda(t)U_2\left(I_2\left(t, \frac{\eta(t)}{\lambda(t)} e^v \right)\right), \quad (t,v) \in [0,T) \times \mathbb{R} \qquad (3.25) \\
&u(T,v) = U_3\left(I_3\left(e^v\right)\right), \qquad v \in \mathbb{R} \qquad (3.26)
\end{aligned}
$$

satisfying the growth condition: for every $\varepsilon > 0$, exists a constant $C(\varepsilon)$ such that $|u(t,v)| \leq C(\varepsilon)e^{\varepsilon v^2}$, for all $v \in \mathbb{R}$.

The process $Y^{(t,y)}(s)$ of (3.1) can be rewritten using the original Brownian motion $W(\cdot)$

$$
Y^{(t,y)}(s) = y \exp\left\{ -\int_t^s r(u) + \eta(u) + \frac{1}{2}||\theta(u)||^2 - \lambda(u)du - \int_t^s \theta'(u)dW_0(u) \right\}
$$

and the dynamics become

$$
dY^{(t,y)}(s) = Y^{(t,y)}(s)\left[-r(s)ds - \eta(s)ds + \lambda(s)ds - \theta'(s)dW(s)\right]
$$

So we compute, by Itô's rule

$$
d\log Y^{(t,y)}(s) = \left[-r(s)ds - \eta(s)ds + \lambda(s)ds - \frac{1}{2}||\theta(s)||^2 ds - \theta'(s)dW(s) \right]
$$

Again by Itô's rule and (3.25) we may compute

$$
\begin{aligned}
&d\left[\exp\left(-\int_t^s \lambda(u)du \right) u(s, \log Y^{(t,y)}(s)) \right] = \\
&- \exp\left(-\int_t^s \lambda(u)du \right) U_1\left(s, I_1\left(s, Y^{(t,y)}(s) \right) \right) ds \\
&- \exp\left(-\int_t^s \lambda(u)du \right) \lambda(s)U_2\left(s, I_2\left(s, \frac{\eta(s)}{\lambda(s)} Y^{(t,y)}(s) \right) \right) ds \\
&- \exp\left(-\int_t^s \lambda(u)du \right) u_v(s, \log Y^{(t,y)}(s))\theta'(s)dW(s)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
& d\left[\bar{F}(s,t)u(s,\log Y^{(t,y)}(s))\right] = \\
& \bar{F}(s,t)U_1\left(s, I_1\left(s, Y^{(t,y)}(s)\right)\right) ds \\
& f(s,t)U_2\left(s, I_2\left(s, \frac{\eta(s)}{\lambda(s)}Y^{(t,y)}(s)\right)\right) ds \\
& \bar{F}(T,t)u_v(s,\log Y^{(t,y)}(s))\theta'(s)dW(s)
\end{aligned}
\tag{3.27}
$$

Defining τ_n as on the proof of lemma (3), integrating from t to τ_n and taking expectations with respect to the original probability measure P, we obtain

$$
\begin{aligned}
u(t,\log y) &= E\Bigg[\int_t^{\tau_n} \bar{F}(s,t)U_1\left(s, I_1\left(s, Y^{(t,y)}(s)\right)\right) \\
&+ f(s,t)U_2\left(s, I_2\left(s, \frac{\eta(s)}{\lambda(s)}Y^{(t,y)}(s)\right)\right) ds \\
&+ \bar{F}(\tau_n,t)u(\tau_n,\log Y^{(t,y)}(\tau_n))\Bigg]
\end{aligned}
$$

The same arguments used on the proof of lemma (3) allow us to let n go to infinity and conclude that

$$u(t,\log y) = G(t,y)$$

We may compute the derivatives of u using the derivatives of G by the chain rule

$$u_t(t,\log y) = G_t(t,y)$$

$$u_y(t,\log y) = yG_y(t,y)$$

$$u_{yy}(t,\log y) = y^2G_{yy}(t,y) + u_y(t,\log y) = y^2G_{yy}(t,y) + yG_y(t,y)$$

After substitution on (3.25,3.26), we obtain (3.23, 3.24). □

Now that we have a Cauchy problem for the function G, we would like to relate it to the function $\mathcal{X}$, and relate the derivatives of the two functions.

We first deduce a useful formula that we will need about the utility functions. From the formula $\tilde{U}(y) = U(I(y)) - yI(y)$, for $0 < y < \infty$, we can take derivatives, and as we assumed that $I(y)$ is differentiable, by the chain rule we have

$$\tilde{U}(y) = U'(I(y))I'(y) - I(y) - I'(y)y = -I(y)$$

So we also have that, for given $0 < z < y < \infty$

$$\begin{aligned} yI(y) - zI(z) - \int_z^y I(\xi)d\xi &= yI(y) - zI(z) + \tilde{U}(y) - \tilde{U}(z) \\ &= U(I(y)) - U(I(z)) \end{aligned}$$

The same identities are valid for utility functions that also depend on time, i.e., for given $t \in [0, T]$ and $0 < z < y < \infty$, we have

$$\tilde{U}(t, y) = U'(t, I(t, y))I'(t, y) - I(t, y) - I'(t, y)y = -I(t, y)$$

which implies

$$yI(t, y) - zI(t, z) - \int_z^y I(t, \xi)d\xi = U(t, I(t, y)) - U(t, I(t, z))$$

With these identities we can prove the following

Lemma 5. *For given $t \in [0, T]$ and $0 < z < y < \infty$ we have*

$$y\mathcal{X}(t, y) - z\mathcal{X}(t, z) - \int_z^y \mathcal{X}(t, \xi)d\xi = G(t, y) - G(t, z) \tag{3.28}$$

$$G_y(t, y) = y\mathcal{X}_y(t, y) \tag{3.29}$$

$$G_{yy}(t, y) = \mathcal{X}_y(t, y) + y\mathcal{X}_{yy}(t, y) \tag{3.30}$$

Proof: The second formula is obtained by differentiating the first one, and the third one is obtained from differentiation of the second one.

To derive the first formula, we fix $t \in [0, T]$ and let $0 < z < y < \infty$. The left hand side of the first formula can be split in three parts, each one corresponding to one of the utility functions that we are considering. We will analyse each of these parts separately for simplicity of writing.

We will use formula (3.5) of the definition of $\mathcal{X}$. The first part, related to the utility function U_1 is

$$E\left[\int_t^T Y^{(t,y)}(s)I_1(s,Y^{(t,y)}(s))\bar{F}(s,t) - Y^{(t,z)}(s)I_1(s,Y^{(t,z)}(s))\bar{F}(s,t)ds\right]$$
$$-\int_z^y E\left[\int_t^T Y^{(t,1)}(s)I_1(s,\xi Y^{(t,1)}(s))\bar{F}(s,t)ds\right]d\xi$$

and changing the order of integration and making a substitution, we can write it as

$$E\left[\int_t^T \bar{F}(s,t)\left(Y^{(t,y)}(s)I_1(s,Y^{(t,y)}(s)) - Y^{(t,z)}(s)I_1(s,Y^{(t,z)}(s)) - \int_{Y^{(t,z)}(s)}^{Y^{(t,y)}(s)} I_1(s,\zeta)d\zeta\right)ds\right]$$

and by what we have observed before, this equals

$$E\left[\int_t^T \bar{F}(s,t)\Big(U_1(s,I_1(s,Y^{(t,y)}(s))) - U_1(s,I_1(s,Y^{(t,z)}(s)))\Big)ds\right] \tag{3.31}$$

The second part, related do the utility function U_2 is

$$E\left[\int_t^T Y^{(t,y)}(s)\frac{\eta(s)}{\lambda(s)}I_2\left(s,\frac{\eta(s)}{\lambda(s)}Y^{(t,y)}(s)\right)\lambda(s)\bar{F}(s,t)\right.$$
$$\left.-Y^{(t,z)}(s)\frac{\eta(s)}{\lambda(s)}I_2\left(s,\frac{\eta(s)}{\lambda(s)}Y^{(t,z)}(s)\right)\lambda(s)\bar{F}(s,t)ds\right]$$
$$-\int_z^y E\left[\int_t^T \lambda(s)Y^{(t,1)}(s)\frac{\eta(s)}{\lambda(s)}I_2\left(s,\frac{\eta(s)}{\lambda(s)}\xi Y^{(t,1)}(s)\right)\bar{F}(s,t)ds\right]d\xi$$

which equals

$$E\left[\int_t^T \overbrace{\bar{F}(s,t)\lambda(s)}^{f(s,t)}\left(Y^{(t,y)}(s)\frac{\eta(s)}{\lambda(s)}I_2\left(s,\frac{\eta(s)}{\lambda(s)}Y^{(t,y)}(s)\right)\right.\right.$$
$$\left.\left.-Y^{(t,z)}(s)\frac{\eta(s)}{\lambda(s)}I_2\left(s,\frac{\eta(s)}{\lambda(s)}Y^{(t,z)}(s)\right) - \int_{Y^{(t,z)}(s)\frac{\eta(s)}{\lambda(s)}}^{Y^{(t,y)}(s)\frac{\eta(s)}{\lambda(s)}} I_1(s,\zeta)d\zeta\right)ds\right]$$

so we have

$$E\left[\int_t^T f(s,t)\left(U_2\left(s, I_2\left(s, \frac{\eta(s)}{\lambda(s)}Y^{(t,y)}(s)\right)\right) - U_2\left(s, I_2\left(s, \frac{\eta(s)}{\lambda(s)}Y^{(t,z)}(s)\right)\right)\right)ds\right] \tag{3.32}$$

Finally, the third part, relative to the utility function U_3 is

$$E\left[Y^{(t,y)}(T)I_3(Y^{(t,y)}(T))\bar{F}(T,t) - Y^{(t,z)}(T)I_3(Y^{(t,z)}(T))\bar{F}(T,t)\right] \\ - \int_z^y E\left[Y^{(t,1)}(T)I_3(\xi Y^{(t,1)}(T))\bar{F}(T,t)\right]d\xi$$

which equals

$$E\left[\bar{F}(T,t)\left(Y^{(t,y)}(T)I_3(Y^{(t,y)}(T)) - Y^{(t,z)}(T)I_3(Y^{(t,z)}(T)) - \int_{Y^{(t,z)}(T)}^{Y^{(t,y)}(T)} I_3(\xi)d\xi\right)\right]$$

and so we have

$$E\left[\bar{F}(T,t)\left(U_3\big(I_3\big(Y^{(t,y)}(T)\big)\big) - U_3\big(I_3\big(Y^{(t,z)}(T)\big)\big)\right)\right] \tag{3.33}$$

Summing (3.31),(3.32) and (3.33) we obtain $G(t,y)-G(t,z)$, and the formula is proved.

□

We can now state the following theorem

Theorem 3 (Hamilton-Jacobi-Bellman equation)**.** *Under the previous assumptions, the value function V is of class $C^{1,2}$ on the set $\{(t,x) \in [0,T) \times \mathbb{R}; x > \mathcal{X}(t,\infty)\}$ and is continuous on $\{(t,x) \in [0,T] \times \mathbb{R}; x > \mathcal{X}(t,\infty)\}$. Furthermore, V satisfies the boundary condition $V(T,x) = U_3(x)$, and satisfies the Hamilton-Jacobi-Bellman equation of dynamic programming on the set $\{(t,x) \in [0,T) \times \mathbb{R}; x > \mathcal{X}(t,\infty)\}$:*

$$V_t(t,x) - \lambda(t)V(t,x) + \max_{\substack{\pi \in \mathbb{R}^N \\ 0 \le c \le \infty \\ 0 \le Z \le \infty}} \left[\frac{1}{2}||\sigma'(t)\pi||^2 V_{xx}(t,x) + U_1(t,c) + \lambda(t)U_2(t,Z) \right. \\ \left. + \Big((r(t)+\eta(t))\,x - c + \pi'\sigma(t)\theta(t) - Z\eta(t)\Big)V_x(t,x)\right] = 0 \tag{3.34}$$

In particular, the value function of the original problem is $V(0,\cdot)$ and the maximization in (3.34) is achieved by the triple $(C(t,x), Z(t,x), \Pi(t,x))$ of (3.17, 3.18 ,3.19).

Proof: We can differentiate the expression $\mathcal{X}(t,\mathcal{Y}(t,x)) = x$ and we obtain

$$\begin{aligned}\mathcal{X}_t(t,\mathcal{Y}(t,x)) + \mathcal{X}_y(t,\mathcal{Y}(t,x))\mathcal{Y}_t(t,x) &= 0\\ \mathcal{X}_t(t,\mathcal{Y}(t,x))\mathcal{Y}_x(t,x) &= 1\end{aligned}$$

On the other hand, by differentiating $V(t,x) = G(t,\mathcal{Y}(t,x))$ we obtain

$$\begin{aligned}V_t(t,x) = G_t(t,\mathcal{Y}(t,x)) + G_y(t,\mathcal{Y}(t,x))\mathcal{Y}_t(t,x)\\ V_x(t,x) = G_y(t,\mathcal{Y}(t,x))\mathcal{Y}_x(t,x) = \mathcal{Y}(t,x)\mathcal{X}_y(t,\mathcal{Y}(t,x))\mathcal{Y}_x(t,x) = \mathcal{Y}(t,x)\\ V_{xx}(t,x) = \mathcal{Y}_x(t,x)\end{aligned}$$

Using these identities, and observing that the condition to obtain the maximum in equation (3.34) decouples into three independent conditions, we have

$$\begin{aligned}G_t(t,\mathcal{Y}(t,x)) &+ G_y(t,\mathcal{Y}(t,x))\mathcal{Y}_t(t,x) - \lambda(t)G(t,\mathcal{Y}(t,x))\\ &+ \max_{\pi\in\mathbb{R}^N} \frac{1}{2}||\sigma'(t)\pi||^2\mathcal{Y}_x(t,x) + \pi'\sigma(t)\theta(t)\mathcal{Y}(t,x)\\ &+ \max_{0\le Z\le\infty} \lambda(t)U_2(t,Z) - Z\eta(t)\mathcal{Y}(t,x)\\ &+ \max_{0\le c\le\infty} U_1(t,c) - c\mathcal{Y}(t,x)\\ &+ (r(t)+\eta(t))\,x\mathcal{Y}(t,x)\end{aligned} \tag{3.35}$$

To find the quantities that maximize this expressions, we set their derivatives equal to zero and obtain respectively

$$\begin{aligned}U_1'(t,c) - \mathcal{Y}(t,x) = 0 \quad&\Rightarrow\quad c = I_1(t,\mathcal{Y}(t,x))\\ \lambda(t)U_2'(t,Z) - \eta(t)\mathcal{Y}(t,x) = 0 \quad&\Rightarrow\quad Z = I_2\left(t, \frac{\eta(t)}{\lambda(t)}\mathcal{Y}(t,x)\right)\\ \sigma(t)\theta(t)\mathcal{Y}(t,x) + \sigma(t)\sigma'(t)\pi\mathcal{Y}_x(t,x) = 0_{\mathbb{R}^N} \quad&\Rightarrow\quad \pi = -(\sigma'(t))^{-1}\theta(t)\frac{\mathcal{Y}(t,x)}{\mathcal{Y}_x(t,x)}\end{aligned}$$

To see that these quantities are indeed maximizers, we compute the second order derivatives

$$\begin{aligned}U_1''(t,c)\\ U_2''(t,Z)\\ \sigma(t)\sigma'(t)\mathcal{Y}(t,x)\end{aligned}$$

We know that both U_1 and U_2 are strictly concave, so their second order derivatives above are strictly negative. Also note that the function $\mathcal{Y}(t,\cdot)$ is strictly decreasing, so $\mathcal{Y}_x(t,\cdot)$ is strictly negative. So as $\sigma(t)\sigma'(t)$ is positive definite, we have that the Hessian matrix of second order derivatives is negative definite. So (3.17, 3.18,3.19) provide the maximization values for c, Z and π.

We now let $y = \mathcal{Y}(t,x)$, and in this case we have $\mathcal{X}(t,y) = x$. Substituting the quantities obtained above in (3.35) and using (3.29) of the previous lemma.

$$
\begin{aligned}
G_t(t,y) \quad &+ \quad y\,\overbrace{\mathcal{X}_y(t,y)\mathcal{Y}_t(t,x)}^{-\mathcal{X}_t(t,y)} - \lambda(t)G(t,y) \\
&- \quad \frac{1}{2}||\theta(t)||^2\frac{y^2}{\mathcal{Y}_x(t,x)} \\
&+ \quad \lambda(t)U_2\left(t, I_2\left(t, \frac{\eta(t)}{\lambda(t)}y\right)\right) - \eta(t)yI_2\left(t, \frac{\eta(t)}{\lambda(t)}y\right) \\
&+ \quad U_1(t, I_1(t,y)) - I_1(t,y)y \\
&+ \quad (r(t)+\eta(t))\,\mathcal{X}(t,y)y
\end{aligned}
$$

By writing the derivative $1/\mathcal{Y}_x = \mathcal{X}_y$ and using (3.23) and formulas (3.29) and (3.30), we have

$$
\begin{aligned}
&-\frac{1}{2}||\theta(t)||^2y^2G_{yy}(t,y) + (r(t)+\eta(t)-\lambda(t))\,yG_y(t,y) \\
&\qquad -y\mathcal{X}_t(t,y) - \frac{1}{2}||\theta(t)||^2y^2\mathcal{X}_y(t,y) \\
&\qquad -I_1(t,y)y - \eta(t)yI_2\left(t, \frac{\eta(t)}{\lambda(t)}y\right) + (r(t)+\eta(t))\,\mathcal{X}(t,y)y \\
= -y\Bigg[&\mathcal{X}_t(t,y) + \frac{1}{2}||\theta(t)||^2y^2\mathcal{X}_{yy}(t,y) + \left(||\theta(t)||^2 - r(t) - \eta(t) + \lambda(t)\right)y\mathcal{X}_y(t,y) \\
&- (r(t)+\eta(t))\,\mathcal{X}(t,y) \\
&+I_1(t,y) + \eta(t)I_2\left(t, \frac{\eta(t)}{\lambda(t)}y\right)\Bigg]
\end{aligned}
$$

By (3.6) this expression is zero, and the theorem is proved. □

The optimal solutions in feedback form given by equations (3.17, 3.18, 3.19), can be written using the value function V and its derivatives

$$
c^*(t,x) = I_1\left(t, V_x(t,x)\right) \tag{3.36}
$$

$$Z^*(t,x) = I_2\left(t, \frac{\eta(s)}{\lambda(s)} V_x(t,x)\right) \tag{3.37}$$

$$\pi(t,x) = -(\sigma'(t))^{-1}\theta(t)\frac{V_x(t,x)}{V_{xx}(t,x)} \tag{3.38}$$

Substituting this formulas in (3.34), we obtain the following partial differential equation

$$\begin{aligned}
V_t(t,x) &- \lambda(t)V(t,x) + (r(t)+\eta(t))\, xV_x(t,x) \\
&- \frac{1}{2}||\theta(t)||^2 \frac{V_x^2(t,x)}{V_{xx}(t,x)} \\
&+ \lambda(t)U_2\left(t, I_2\left(t, \frac{\eta(t)}{\lambda(t)}V_x(t,x)\right)\right) - \eta(t)I_2\left(t, \frac{\eta(t)}{\lambda(t)}V_x(t,x)\right)V_x(t,x) \\
&+ U_1(t, I_1(t, V_x(t,x))) - I_1(t, V_x(t,x))V_x(t,x) = 0
\end{aligned}$$

The Hamilton-Jacobi-Bellman, in the general case, doesn't have a solution in the classical sense, and several concepts of generalization of its solution have been developed, the most celebrated being, possibly, the *viscosity solution*. The importance of the Hamilton-Jacobi-Bellman equation is due to its intimate connection with the optimal solution, as we have shown above. It is also important to note that there are several numerical methods to approximate the solution V, including in situations where there are constraints. For this topics we suggest [5].

3.3 The family of discounted CRRA utilities

3.3.1 Discounted CRRA Power utilities

We now derive the optimal strategies for the case where the agent's utility functions for consumption, size of the legacy and size of terminal wealth are all discounted CRRA power utilities, given by

$$U_1(t,c) = e^{-\rho t}\frac{c^\gamma}{\gamma}, \quad U_2(t,Z) = e^{-\rho t}\frac{Z^\gamma}{\gamma} \quad U_3(X) = e^{-\rho T}\frac{X^\gamma}{\gamma} \tag{3.39}$$

where the risk aversion [4] parameter γ verifies $\gamma < 1$ and $\gamma \neq 0$, and the discount rate ρ is positive.

Differentiating the previous utility functions with respect to their second variable and inverting, we obtain:

$$I_1(t,y) = I_2(t,y) = e^{\rho t/(\gamma-1)} y^{1/(\gamma-1)}, \quad I_3(y) = e^{\rho T/(\gamma-1)} y^{1/(\gamma-1)}$$

By using this functions we have $\mathcal{X}(y) = \mathcal{X}(1)y^{1/(\gamma-1)}$, so $\mathcal{Y}(x) = \left(\frac{x}{\mathcal{X}(1)}\right)^{\gamma-1}$

And we can write

$$c^*(t) = e^{\rho t/(\gamma-1)}\left(\frac{x+b(0)}{\mathcal{X}(1)}\right)\left(\frac{\bar{H}_0(t)}{\bar{F}(t)}\right)^{1/(\gamma-1)} \tag{3.40}$$

$$Z^*(t) = e^{\rho t/(\gamma-1)}\left(\frac{x+b(0)}{\mathcal{X}(1)}\right)\left(\frac{\eta(t)}{\lambda(t)}\frac{\bar{H}_0(t)}{\bar{F}(t)}\right)^{1/(\gamma-1)} \tag{3.41}$$

$$\xi^* = e^{\rho T/(\gamma-1)}\left(\frac{x+b(0)}{\mathcal{X}(1)}\right)\left(\frac{\bar{H}_0(T)}{\bar{F}(T)}\right)^{1/(\gamma-1)} \tag{3.42}$$

$$\tag{3.43}$$

Substituting these into (2.21) and adding $b(t)$ we obtain

[4] We assume that the risk aversion coefficient is the same in the three utility functions, otherwise it would be impossible to obtain closed form solutions.

$$X^*(t)+b(t)=\frac{1}{\bar{H}_0(t)}E\left[\int_t^T \bar{H}_0(t)(c^*(u)+\eta(u)Z^*(u))du+\bar{H}_0(T)\xi^*\middle|\mathcal{F}(t)\right]=$$

$$=\frac{x+b(0)}{\mathcal{X}(1)\bar{H}_0(t)}E\left[\int_t^T e^{\rho u/(\gamma-1)}\frac{\bar{H}_0(u)^{\gamma/(\gamma-1)}}{\bar{F}(u)^{1/(\gamma-1)}}K(u)du+\right.$$

$$\left.+\,e^{\rho T/(\gamma-1)}\frac{\bar{H}_0(T)^{\gamma/(\gamma-1)}}{\bar{F}(T)^{1/(\gamma-1)}}\middle|\mathcal{F}(t)\right]$$

where

$$K(u)=1+\frac{\eta(u)^{\gamma/(\gamma-1)}}{\lambda(u)^{1/(\gamma-1)}}$$

We define the martingale

$$\Lambda(t)=\exp\left\{\frac{\gamma}{1-\gamma}\int_0^t\theta'(u)dW(u)-\frac{\gamma^2}{2(1-\gamma)^2}\int_0^t||\theta(u)||^2du\right\}$$

and we also define

$$H(t)=\frac{\lambda(t)+\rho}{1-\gamma}-\frac{\gamma}{2(1-\gamma)^2}||\theta(t)||^2-\frac{\gamma}{1-\gamma}(r(t)+\eta(t))$$

We observe that $H(\cdot)$ is deterministic, and we have, for all $t\in[0,T]$

$$e^{\rho t/(\gamma-1)}\frac{\bar{H}_0(t)^{\gamma/(\gamma-1)}}{\bar{F}(t)^{1/(\gamma-1)}}=\exp\left\{-\int_0^t H(s)ds\right\}\Lambda(t)$$

Using the fact that $\Lambda(\cdot)$ is a martingale, we can write

$$X^*(t)+b(t)=\frac{x+b(0)}{\mathcal{X}(1)\bar{H}_0(t)}E\left[\int_t^T K(u)\Lambda(u)\exp\left\{\int_0^u H(s)ds\right\}du+\right.$$

$$\left.+\exp\left\{\int_0^T H(s)ds\right\}\Lambda(T)\middle|\mathcal{F}(t)\right]$$

$$=\frac{x+b(0)}{\mathcal{X}(1)\bar{H}_0(t)}\Lambda(t)N(t)$$

where

$$N(t)=\int_t^T K(u)\exp\left\{-\int_0^u H(s)ds\right\}du+\exp\left\{-\int_0^T H(u)du\right\}$$

Solving the previous equation for $\frac{x+b(0)}{\mathcal{X}(1)}$, and substituting in (3.40) and (3.41), we obtain

$$c^*(t) = \frac{X^*(t)+b(t)}{N(t)} \exp\left\{-\int_0^t H(s)ds\right\} = \frac{1}{e(t)}(X^*(t)+b(t)) \tag{3.44}$$
$$Z^*(t) = d(t)(X^*(t)+b(t)) \tag{3.45}$$

where

$$e(t) = \exp\left\{-\int_t^T H(u)du\right\} + \int_t^T K(u)\exp\left\{-\int_t^u H(s)ds\right\}du$$

and

$$d(t) = \left(\frac{\eta(t)}{\lambda(t)}\right)^{1/(\gamma-1)} \frac{1}{e(t)}$$

and from (3.45) and (2.24) we obtain

$$p^*(t) = \eta(t)(Z^*(t) - X^*(t)) = \eta(t)((d(t)-1)X^*(t) + d(t)b(t)) \tag{3.46}$$

Substituting (3.40) and (3.41) in (2.23), we have

$$\begin{aligned} M(t) = \frac{x+b(0)}{\mathcal{X}(1)} E\Bigg[\int_0^T K(u)\exp\left\{-\int_0^u H(s)ds\right\}\Lambda(u)du \\ + \exp\left\{-\int_0^T H(u)du\right\}\Lambda(T)\Bigg|\mathcal{F}(t)\Bigg] - E\left[\int_0^T i(u)\bar{H}_0(u)du\Bigg|\mathcal{F}(t)\right] = \\ = \frac{x+b(0)}{\mathcal{X}(1)}\left(\Lambda(t)N(t) + \int_0^t K(u)\exp\left\{-\int_0^u H(s)ds\right\}\Lambda(u)du\right) \\ -E\left[\int_0^T i(u)\bar{H}_0(u)du\Bigg|\mathcal{F}(t)\right] \end{aligned}$$

$$\begin{aligned} dM(t) &= \frac{x+b(0)}{\mathcal{X}(1)} N(t)d\Lambda(t) - dE\left[\int_0^T i(u)\bar{H}_0(u)du\Bigg|\mathcal{F}(t)\right] \\ &= (X^*(t)+b(t))\bar{H}_0(t)\frac{\gamma}{1-\gamma}\theta'(t)dW(t) + b(t)\bar{H}_0(t)\theta'(t)dW(t) \end{aligned}$$

Having the integrand in the stochastic representation of the martingale M, we can now substitute it in (2.22)

$$\pi(t) = (\sigma'(t))^{-1}(X^*(t)+b(t))\frac{1}{1-\gamma}\theta(t) \tag{3.47}$$

These results coincide with the ones obtained through dynamic programming in [1] in the case where the market is complete. For the economic interpretations of these results we refer to the same paper.

3.3.2 Discounted Logarithmic CRRA utilities

We now derive the optimal strategies for the case where the agent's utility functions for consumption, size of the legacy and size of terminal wealth are all discounted CRRA logarithmic utilities, given by

$$U_1(t,c) = e^{-\rho t}\log c, \quad U_2(t,Z) = e^{-\rho t}\log Z \quad U_3(X) = e^{-\rho T}\log X \tag{3.48}$$

We observe that this logarithmic utilities can be seen as the limit case of (3.39), when we make γ go to zero. Indeed, if we fix c, t and ρ in U_1 we have by L'Hôpital's rule

$$\lim_{\gamma\to 0} e^{-\rho t}\frac{c^\gamma}{\gamma} = \lim_{\gamma\to 0} e^{-\rho t}\frac{e^{\gamma\log c}}{\gamma} = \lim_{\gamma\to 0} e^{-\rho t}\log c e^{\gamma\log c} = e^{-\rho t}\log c \tag{3.49}$$

By inverting the derivatives of (3.48), we obtain

$$I_1(t,y) = I_2(t,y) = \frac{e^{-\rho t}}{y}, \quad I_3(y) = \frac{e^{-\rho T}}{y}$$

Substituting in the definition of $\mathcal{X}(\cdot)$, we obtain

$$\mathcal{X}(y) = \frac{1}{y}E\left[\int_0^T e^{-\rho t}\bar{F}(t)(1+\lambda(t))dt + e^{-\rho T}\bar{F}(T)\right] = \frac{1}{y}\mathcal{X}(1)$$

We have

$$c^*(t) = e^{-\rho t}\frac{\bar{F}(t)}{\mathcal{Y}(x+b(0)\bar{H}_0(t)} \tag{3.50}$$

$$Z^*(t) = e^{-\rho t}\frac{\lambda(t)\bar{F}(t)}{\mathcal{Y}(x+b(0)\bar{H}_0(t)\eta(t)} \tag{3.51}$$

$$\xi^* = e^{-\rho T}\frac{\bar{F}(T)}{\mathcal{Y}(x+b(0)\bar{H}_0(T)} \tag{3.52}$$

Substituting these into (2.21) and adding $b(t)$ we obtain

$$\begin{aligned}
X^*(t)+b(t) &= \frac{1}{\bar{H}_0(t)}E\left[\int_t^T \bar{H}_0(t)(c^*(u)+\eta(u)Z^*(u))du + \bar{H}_0(T)\xi^*\middle|\mathcal{F}(t)\right] \\
&= \frac{1}{\mathcal{Y}(x+b(0))\bar{H}_0(t)}\left(\int_t^T e^{-\rho u}\bar{F}(u)(1+\lambda(u))du + e^{-\rho T}\bar{F}(T)\right) \\
&= \frac{1}{\mathcal{Y}(x+b(0))\bar{H}_0(t)}a(t)
\end{aligned}$$

where

$$a(t) = \int_t^T e^{-\rho u}\bar{F}(u)(1+\lambda(u))du + e^{-\rho T}\bar{F}(T)$$

Solving the previous equation for $\mathcal{Y}(x+b(0))\bar{H}_0(t)$ and substituting in the optimal strategies, we obtain

$$c^*(t) = e^{-\rho t}\frac{\bar{F}(t)}{a(t)}(X^*(t)+b(t)) \tag{3.53}$$

$$Z^*(t) = e^{-\rho t}\frac{\lambda(t)\bar{F}(t)}{\eta(t)a(t)}(X^*(t)+b(t)) \tag{3.54}$$

and from (3.54) and (2.24) we obtain

$$p^*(t) = \eta(t)(Z^*(t) - X^*(t)) = \eta(t)((g(t)-1)X^*(t) + g(t)b(t)) \tag{3.55}$$

where

$$g(t) = e^{-\rho t}\frac{\lambda(t)\bar{F}(t)}{\eta(t)a(t)}.$$

Substituting (3.50) and (3.51) in (2.23) we obtain

$$M(t) = \frac{1}{\mathcal{Y}(x+b(0))}a(0) - E\left[\int_0^T i(u)\bar{H}_0(u)du|\mathcal{F}(t)\right]$$

and we have the differential

$$dM(t) = \bar{H}_0(t)b(t)\theta'(t)dW(t) \tag{3.56}$$

Having the integrand in the stochastic representation of the martingale M, we can now substitute it in (2.22), and obtain the optimal portfolio

$$\pi(t) = (\sigma'(t))^{-1}\theta(t)(X^*(t)+b(t)) \tag{3.57}$$

Observe that if we consider $\gamma = 0$ in auxiliary functions H, K, e and d we defined when deriving the solutions for CRRA Power utilities, they become

$$H(t) = \lambda(t) + \rho$$

$$K(t) = 1 + \lambda(t)$$

$$e(t) = e^{-\rho(T-t)}\bar{F}(T,t) + \int_t^T 1 + \lambda(u)e^{-\rho(u-t)}\bar{F}(u,t)du$$

$$d(t) = \frac{\lambda(t)}{\eta(t)}\frac{1}{e(t)}$$

Recalling that $\bar{F}(s,t) = \bar{F}(s)/\bar{F}(t)$, we have

$$e^{\rho t}\frac{a(t)}{\bar{F}(t)} = e^{-\rho(T-t)}\frac{\bar{F}(T)}{\bar{F}(t)} + \int_t^T (1 + \lambda(u))e^{-\rho(u-t)}\frac{\bar{F}(u)}{\bar{F}(t)}du = e(t)$$

and

$$e^{\rho t}\frac{\eta(t)}{\lambda(t)}\frac{a(t)}{\bar{F}(t)} = \frac{1}{d(t)}$$

Finally, if we make $\gamma = 0$ in (3.47) we obtain

$$\pi(t) = (\sigma'(t))^{-1}(X^*(t) + b(t))\theta(t)$$

So we conclude that the optimal solutions for CRRA logarithmic utilities are obtained by letting $\gamma = 0$ in the optimal solutions for the CRRA power utilities.

Chapter 4

Conclusions and future work

We considered an optimal consumption, life insurance purchase and investment problem in a general Brownian financial market model with stochastic coefficients. We have used tools from convex analyses to obtain solutions for general utility functions. In the special case of deterministic coefficients, we deducted the Hamilton-Jacobi-Bellman equation of dynamic programming, and obtained closed form solutions for the case of CRRA utilities, and those solution coincide with the ones obtained from the dynamical programming approach.

It should be noted that, in a certain way, this work investigated how the behaviour of optimal strategies changes with the utility functions that are considered, and we have obtained that, in general, the optimal solutions depend on the inverse of the derivative of the utility functions. For future work, it would be interesting to investigate the behaviour of the optimal solutions when there are changes in the underlying financial market using tools from Malliavin calculus.

It would also be interesting to introduce other features to the model, namely, changing the form of the functional, considering that the life insurance contract finishes at time T, and that from time T, he stops receiving income, but that he still consumes and invests on the market after time T, up until his death. Other future working topic is to extend the model to include the possibility that the agent may stablish a contract, such as a loan contract (for buying car, house, etc.) and a pension contract with a bank.

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